Assignment 3

Due Friday, 7/11/2014

Remember to show your work for credit!

Problem 1

Solve each of the following linear, second order equations. If initial conditions are given, solve for any constants.

(a) $4y'' - y = 0$, $y(-2) = 1$, $y'(-2) = -1$.

(b) $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$.

(c) $6y'' - y' - y = 0$.

(d) $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = 1/2$.

(e) $y'' - 9y' - 9y = 0$.

Problem 2

Find the general solution for each of the following first order equations. Use any applicable method.

(a) $(e^x + 1)\, y' = y - ye^x$.

(b) $y' = 2e^{x+y}$.

(c) $y' + y = 1/\,(1 + e^x)$.

Problem 3

The principle of superposition says that if $y_1(t)$ and $y_2(t)$ are solutions to a second order linear equation, then $y(t) = C_1 y_1(t) + C_2 y_2(t)$ is also a solution. (We proved this in class.)

Show that $y_1(t) = 1$ and $y_2(t) = \sqrt{t}$ are both solutions to the differential equation

$$yy'' + (y')^2 = 0$$

for $t > 0$. Are y_1 and y_2 linearly independent solutions (hint: use the Wronskian)? Show that $C_1 + C_2\sqrt{t}$ is not, in general, a solution to the differential equation. Does this violate the principle of superposition? Why or why not?

Problem 4

In class we discussed the logistic model with constant effort harvesting, modeled by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right) - hN,$$

where $N(t)$ was the abundance of fish, $K > 0$ was the carrying capacity, $r > 0$ was the maximum per capita growth rate and $h > 0$ was a measure of fishing effort. This model assumes that the per capita growth rate of fish is r for very low densities and decreases linearly with density.

This model is often not realistic, because growth rates may be lower at very low densities (e.g., if it is too hard to find a mate when the population is very small). To incorporate this idea (known as an Allee effect) into our model, we will assume that the per capita birth rate is modeled by

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(\frac{N}{K_0} - 1\right)\left(1 - \frac{N}{K}\right),$$

where $0 < K_0 < K$. For the rest of the model, we will assume that $K_0 = .1$ and $K = 1$, so our model (including harvesting) will be

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(10N - 1\right)\left(1 - N\right) - hN.$$

(a) Sketch the growth term and harvest term for different values of h (hint: look at figure 3 in the notes on the logistic harvest model). Be sure to include examples of both small and large h. What does this plot tell us about the fixed points of our model? How many are there? Are they stable or unstable? Does this depend on h?

(b) Find a formula for the fixed points (in terms of r and h). Find the critical value of h as a function of r (i.e., the value where the number of fixed points changes)? Do these results match those from part (a)?

(c) Fix $r = 1$. Make a bifurcation diagram for this model. Use solid lines to indicate stable fixed points and dashed lines to indicate unstable fixed points.

(d) Suppose that h starts slightly below the critical value you found in part (b) and the population is at the positive stable equilibrium. What would happen if h increased to just above the critical value?