Assignment 7
Due Wednesday, 8/20/2014

Problem 1

Solve the given linear system for the vector $\boldsymbol{x}$. Find all solutions.

(a) $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 5 \\ -15 \end{pmatrix}$.

(b) $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Problem 2

Find all eigenvalues and eigenvectors of the given matrix.

(a) $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Problem 3

Find the general solution of each of the following systems of differential equations, then use the given initial conditions to solve for any constants.

(a) $\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x}$, with $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) $\boldsymbol{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \boldsymbol{x}$, with $\boldsymbol{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) $\boldsymbol{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \boldsymbol{x}$, with $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Problem 4

Consider the system of differential equations

$$\boldsymbol{x}' = A\boldsymbol{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and a, b, c and d are constants.

(a) Find the eigenvalues of A (in terms of a, b, c and d).

(b) Let $T = a + d$ and $D = ad - bc$. (These are called the trace and determinant of A, respectively.) Find an equation for the eigenvalues of A in terms of only T and D.

(c) For what values of T and D are the eigenvalues of A both real and positive (i.e., the solution to our ODE is an unstable node)? For what values are the eigenvalues both real and negative (i.e., the solution is a stable node)? For what values are the eigenvalues complex with positive real part (i.e., the solution is an unstable spiral)? For what values are the eigenvalues complex with negative real part (i.e., the solution is a stable spiral)? For what values is one eigenvalue positive and the other negative (i.e., the solution is a saddle)?

(d) Make a graph on the T-D plane (i.e., with T on the x-axis and D on the y-axis) with each of the regions you found in part c marked.

(e) Without finding any eigenvalues, state the solution type for the ODE

$$\boldsymbol{x}' = \begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix} \boldsymbol{x}.$$