Extra Credit Assignment

Due 8/22/2014

This assignment is worth up to 5 points on your final grade. In addition, if you complete every problem, I will give you the option of replacing your midterm grade with the average of your midterm and final. That is, you can recover up to half of the points lost on your midterm.

These problems are not meant to be easy. You may work with anyone else from our class, but your write-up should be your own.

Problem 1 Resonance

Consider an undamped spring mass system with unit mass and unit spring constant. Suppose that the external forces on this spring alternate between a constant positive force (pulling on the spring out) and a constant negative force (pushing the spring in). After some number of pushes and pulls, the external force will remain constant. This can be modeled by

$$y'' + y = f_n(t), \tag{1}$$

where

$$f_n(t) = u(t) + 2\sum_{k=1}^{n}(-1)^k u(t - k\pi).$$

(a) Graph $f_3(t)$ and $f_4(t)$ on the interval $0 \le t \le 6\pi$ (use two separate graphs).

(b) For $n = 15$, solve (1). Plot the solution for $0 \le t \le 60$. Why does the solution behave like it does?

(c) What happens to the solution as n increases? What happens as $n \to \infty$?

(d) Compare these results with those for the IVP

$$y'' + y = \cos(t), \quad y(0) = 0, \;\; y'(0) = 0.$$

Why do you think these problems are related?

Problem 2 Perturbation Methods

If an object is thrown vertically into the air, its height can be modeled by the initial value problem

$$y''(t) + \epsilon y'(t) + 1 = 0, \quad \text{with} \quad y(0) = 0, \quad y'(0) = 1. \tag{2}$$

Here, y is the height at time t and ϵ is a measure of air friction. Since air friction is typically very small, we can assume that $0 < \epsilon \ll 1$.

(a) First, solve the IVP when $\epsilon = 0$. Plot the solution.

(b) Show that

$$y(t) = \frac{(1+\epsilon)}{\epsilon^2}\left(1 - e^{-\epsilon t}\right) - \frac{t}{\epsilon} \tag{3}$$

is the solution to (2) when $\epsilon \neq 0$.

(c) Find the limit as ϵ goes to 0 of (3).

(d) Now try a solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n(t) \cdot \epsilon^n,$$

where each a_n is a function that depends on t but not ϵ. (This is a Taylor expansion in ϵ rather than t.)

Show that $a_n(0) = 0$ for every n, that $a_0'(0) = 1$ and that $a_n'(0) = 0$ for all $n > 1$.

(e) If we only write out the first 3 terms of this series solution, we have

$$y(t) = a_0(t) + a_1(t)\epsilon + a_2(t)\epsilon^2 + a_3(t)\epsilon^3 + \cdots$$

Substitute this guess in (2) and combine all terms with equal powerss of ϵ. You should get an equation of the form

$$A(t) + B(t)\epsilon + C(t)\epsilon^2 + D(t)\epsilon^3 + \cdots = 0.$$

By setting each of these coefficients to zero, solve for $a_0(t)$, $a_1(t)$ and $a_2(t)$.

(f) Find the Taylor expansion of (3) about $\epsilon = 0$. Show that the first three terms are the same as those found in part (e).

Problem 3 Reduction of Order

Consider the differential equation

$$x^2y'' - x(x+4)y' + (2x+6)y = x^4e^x.$$

(a) Check that $y_1 = x^2$ is a solution of the homogeneous problem.

(b) Find the general solution to the homogeneous problem.

(c) Find the general solution to the nonhomogeneous problem.

(d) Suppose the initial conditions $y(0) = y'(0) = 0$ are given. Can you solve the corresponding initial value problem? Why or why not?

Problem 4 Nonlinear Pendulum

Suppose we have a pendulum with a mass on the end and that the only forces acting on the system are gravity and friction. Let $y(t)$ be the angle between the pendulum and the downward vertical direction. For small displacements (i.e., the mass does not move far from its resting position), this can be modeled by the differential equation

$$y'' + \gamma y' + ky = 0,$$

which we studied in class. However, if the mass moves far enough, then this approximation is no longer valid. Instead, we must use the differential equation

$$y'' + \gamma y' + k\sin(y) = 0. \tag{4}$$

(a) Let $u(t) = y'$ and $v(t) = y$. Write (4) as a system of two first order equations.

(b) Find all the equilibria of this system. Linearize the system about each of these equilibria and determine the type of each (e.g., stable or unstable node, stable or unstable focus, saddle).

(c) Consider the function

$$V(u, v) = \frac{u^2}{2} - k\cos(v). \tag{5}$$

When $\gamma = 0$, show that $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$. This means that solutions must stay on level sets of V. Plot several level sets of V (on the same graph) and use this plot to sketch some solutions of (4).

(d) What are the critical points of V? (i.e., where are $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ equal to 0?)

(e) When $\gamma > 0$, show that $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$. For what values of u and v is $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$? For what values of u and v is $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$.