Homework 4

Due Friday, July 21 2017

Problem 1 (5 points)

Suppose that you perform the following experiment: At 5pm, you place a single atom of some radioactive substance in a container on your desk. (Let's not worry about how we move or even find a single atom.) This atom is known to decay with a rate constant of $\lambda = 0.001$ per second. In the morning, you check on your atom and find that it did not decay overnight. You then watch it for one more hour and record whether or not it decays while you watch.

(a) If you get to the office at 8am, what is the probability that your atom decays in the hour you watch it?

(b) Now suppose that you arrive an hour late, and so you watch your atom from 9am to 10am. What is the probability that your atom decays in the hour you watch it?

(c) In contrast, suppose that you are watching your friend run a marathon and that this friend can finish a marathon, on average, in 4 hours. If you show up at the finish line 3 hours into the race and you know that your friend hasn't finished yet, you have a certain probability of seeing them finish in the next hour. If you instead show up at the finish line 4 hours into the race and you know that your friend hasn't finished yet, do you have a higher or lower probability of seeing them finish in the next hour? Why is this different from the radioactive decay example?

Problem 2 (10 points)

Suppose that you have N radioactive atoms on your desk. Assume that the decay time of each atom is exponentially distributed with rate constant λ. That is, if T_i is the (random) time at which the ith atom decays, then

$$\Pr\left[T_i > t\right] = e^{-\lambda t}, \tag{1}$$

where t is the amount of time since you placed all the atoms on your desk. Furthermore, assume that all the decay times are independent. Define

$$T^* = \min\{T_1, T_2, \ldots, T_N\} \tag{2}$$

be the (random) time at which the first atom decays. Find the cumulative distribution function (cdf) and probability density function (pdf) of T^*. What is the expected value of T^*? What is the standard deviation of T^*? (Remember, the standard deviation of a random variable X is defined as $\sigma(X) = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$, where $\langle \cdot \rangle$ denotes the expected value of a random variable.)

Problem 3 (10 points)

Suppose that you start the day with N radioactive atoms on your desk, and that the time to decay T for each atom is identically and independently distributed, with

$$\Pr[T > t] = e^{-\lambda t}. \tag{3}$$

Let T_n be the (random) amount of time that you have to wait until exactly n of the atoms have decayed. Find the cdf and pdf of T_n for each $n = 0, 1, \ldots, N$. Find the expected value of T_N.

Extra Credit: Find the expected value of T_n for arbitrary $n = 0, 1, \ldots, N$.

Problem 4 (15 points)

Many rare events can be modeled with a similar probabilistic description. For example, let $N(t)$ be the number of large floods that will occur in Phoenix in the next t years. From our perspective, this is essentially a random phenomenon, so we will define

$$P_n(t) = \Pr[N(t) = n], \tag{4}$$

for every $n = 0, 1, \ldots$. That is, $P_n(t)$ is the probability that there will be exactly n large floods in the next t years.

We will assume that P_0 has the memoryless property discussed in class. That is, the chance of not having a flood next year is independent of the floods from this year. To be more precise,

$$P_0(t+s) = P_0(t)P_0(s) \tag{5}$$

Furthermore, we will assume that, over a short enough interval of time Δt, the probability of two floods both happening in the same interval is approximately zero. That is, we don't expect to have two 100-year floods in the same week. This means that

$$P_{n+1}(t + \Delta t) = P_{n+1}(t)P_0(\Delta t) + P_n(t)P_1(\Delta t). \tag{6}$$

In addition, we will assume that the probability of one flood happening in a short time span is roughly proportional to the length of time, so that

$$P_1(\Delta t) = \lambda \Delta t \tag{7}$$

and

$$P_0(\Delta t) = 1 - \lambda \Delta t + o(\Delta t). \tag{8}$$

(Remember that $o(x)$ means that $\lim_{x \to 0} o(x)/x = 0$.)

(a) Show that, in the limit as $\Delta t \to 0$,

$$\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} = -\lambda P_0(t) \tag{9}$$

with $P_0(0) = 1$, and

$$\frac{\mathrm{d}P_{n+1}(t)}{\mathrm{d}t} = \lambda P_n(t) - \lambda P_{n+1}(t) \tag{10}$$

with $P_{n+1}(0) = 0$.

(b) Solve (9) for $P_0(t)$. Use this solution to solve (10) for $P_1(t)$ and use that solution to solve (10) for $P_2(t)$.

(c) Find $P_n(t)$ for all n. (The easiest way to do this is to find a pattern in the first few P_n, then guess the formula. To show that your guess is correct, suppose that the formula you found holds for some integer $n = k$, then use (10) to show that it also holds for $n = k + 1$.)

(d) If $\lambda = 0.01$, the floods are called "100 year floods". What is the probability that there is exactly one 100 year flood in the next one hundred years? What is the probability that there are no 100 year floods in the next one hundred years? What is the probability that there is at least one 100 year flood in the next one hundred years?