Homework 6

Due Friday, August 11 2017

Problem 1 (10 points)

Suppose that we have a linear system of differential equations given by

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2, \\ \dot{x_2} = cx_1 + dx_2. \end{cases} \tag{1}$$

Writing this as a matrix equation, we have $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

Recall that the trace of a matrix is the sum of its diagonal entries, so $\mathrm{Tr}(A) = a + d$ and that the determinant of a 2×2 matrix is $\det(A) = ad - bc$.

(a) Find the eigenvalues of A in terms of $\mathrm{Tr}(A)$ and $\det(A)$. (You should not have any a's, b's, c's or d's in your answer.)

(b) Find conditions on $\mathrm{Tr}(A)$ and $\det(A)$ such that the equilibrium of (1) is a stable node, an unstable node, a stable spiral, an unstable spiral or a saddle. For instance, you should find that the equilibrium is a saddle if and only if $\det(A) < 0$.

(c) Make a graph with $\mathrm{Tr}(A)$ on the x-axis and $\det(A)$ on the y-axis and mark the regions where (1) has each type of equilibrium.

Problem 2 (15 points)

Suppose that we have two species of herbivores in the same region competing with each other for resources. We will let $N(t)$ represent the population density of one of the species at time t and let $M(t)$ represent the population of the other species at time t. We will assume that each species would grow logistically in the absence of competition. That is, if $M(t) = 0$, then we would have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r_N N \left(1 - \frac{N}{K_N}\right), \tag{3}$$

where $r_N, K_N > 0$ are constants. Similarly, if $N(t) = 0$, then we would have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = r_M M \left(1 - \frac{M}{K_M}\right), \tag{4}$$

where $r_M, K_M > 0$ are constants. However, if the two species are both present, then both of their growth rates are reduced. We will assume that the reduction in per-capita growth rate is proportional to the population density of the competing species. This means that

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= r_N N \left(1 - \frac{N}{K_N} - \frac{\alpha M}{K_N}\right), \\ \frac{\mathrm{d}M}{\mathrm{d}t} &= r_M M \left(1 - \frac{M}{K_M} - \frac{\beta N}{K_M}\right), \end{aligned} \tag{5}$$

where $\alpha, \beta > 0$. For ease of calculation, we will assume that $\alpha\beta \neq 1$.

This is called the Lotka-Volterra competition model.

(a) Show that if we let $x = N/K_N$, $y = M/K_M$, $\tau = r_N t$, $r = r_M/r_N$, $a = \alpha K_M/K_N$ and $b = \beta K_N/K_M$, we can rewrite (5) as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= x\left(1 - x - ay\right), \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= ry\left(1 - y - bx\right). \end{aligned} \tag{6}$$

(b) Find all four of the equilibria of (6). (They may depend on the parameters a, b and r.) Three of these should be fairly easy, and the fourth is somewhat more complicated. Under what conditions is there a positive equilibrium? (That is, when is there an equilibrium (x^*, y^*) with both $x^* > 0$ and $y^* > 0$?)

(c) Find the Jacobian of this system. That is, if $f_1(x, y) = x(1 - x - ay)$ and $f_2(x, y) = ry(1 - y - bx)$, find $Df(x)$.

(d) Use the Jacobian from part (c) to find the linearization of (6) about each equilibrium you found in part (b). Classify each equilibrium as a stable/unstable node, a stable/unstable spiral or a saddle. Since x and y represent populations, you do not need to classify any equilibrium with $x^* < 0$ or $y^* < 0$. Is there any circumstance where the positive equilibrium is stable? (**Update**: If the positive equilibrium is stable, you do not need to determine whether it is a node or spiral, but doing so will be worth some extra credit.)

Problem 3 (15 points)

Consider the model of a pendulum that we discussed in class:

$$\ddot{\theta} = -a\dot{\theta} - b\sin\theta. \tag{7}$$

As in class, we will let $x_1 = \theta$ and $x_2 = \dot{\theta}$, so we have the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -ax_2 - b\sin x_1 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}. \tag{8}$$

We already found that this system has equilibria at $(x_1^*, x_2^*) = (n\pi, 0)$, where n is any integer. Moreover, we found that the linearized system is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b\cos(n\pi) & -a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{9}$$

where $y_1 = x_1 - n\pi$ and $y_2 = x_2$.

(a) If our pendulum is frictionless, then $a = 0$. What can you conclude about the stability and type of each equilibrium in this case? (You should look at the cases where n is even and where n is odd separately.)

(b) Since this is a physical system, we can calculate its energy (kinetic energy plus potential energy). After taking into account all the changes of variables we made, we find that the energy is

$$E(x_1, x_2) = \frac{1}{2}(x_2)^2 - b\cos(x_1). \tag{10}$$

Draw the level sets of E. That is, on a graph with x_1 on the x-axis and x_2 on the y-axis, plot $E(x_1, x_2) = C$ for various values of C.

(c) Show that if $a = 0$ the energy is conserved. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(x_1(t), x_2(t)) = 0. \tag{11}$$

Using this result and your graph from part (b), sketch the phase portrait of (8) when $a = 0$.

(d) Show that if $a > 0$ then the energy is strictly decreasing (unless the pendulum is at equilibrium). That is, show that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(x_1(t), x_2(t)) \leq 0, \tag{12}$$

with equality only when $x_1(t) = n\pi$ and $x_2(t) = 0$. What does this tell you about the phase portrait of (8) when $a > 0$?