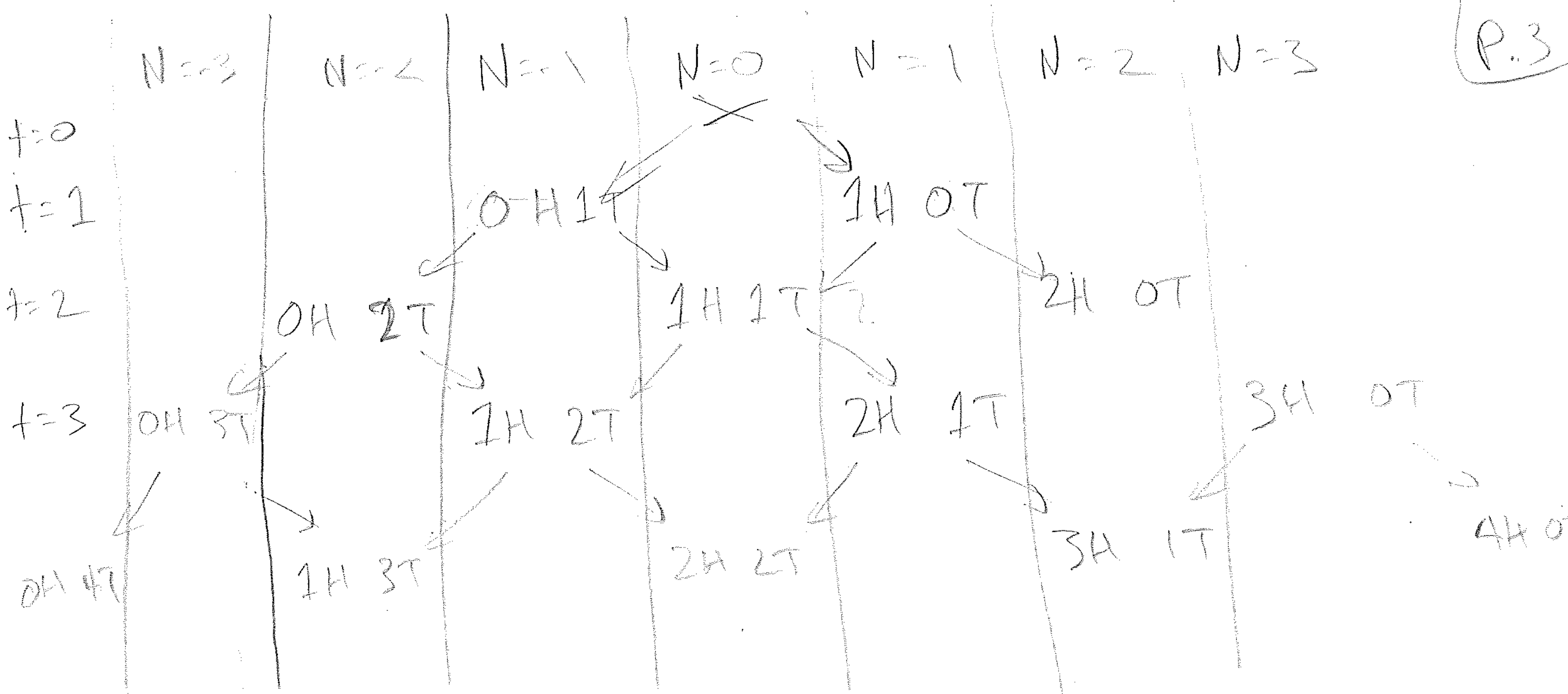
Simple Random Walk:

Last time, we talked about the exponential decay process. That model has many merits, but one fairly important flaw: the number of remaining atoms can only go down, not up. (Alternatively, the number of decay events can only increase.) Ultimately, we would like to model things like reversible chemical reactions or population growth, but the variables of interest in those problems can both increase and decrease. Today, we will look at one of the simplest models that fits these criteria: the simple random walk.

Suppose that you are playing a particularly simple game at the casino. The dealer flips a (possibly biased) coin, then you win one dollar if it ends up heads and lose one dollar if it ends up tails. We want to know how much money you will have after t flips. Since this is a random process, the best we can do is find

$$P_n(t) = \Pr[N(t) = n],$$

where $N(t)$ is the number of dollars you have at time t. (For our purposes, you are allowed to go into debt, so n can be negative. Assume that you start with 0 dollars, ~~so~~ $P_0(0) = 1$ and $P_n(0) = 0$ for all other n. Furthermore, assume that the coin comes up heads with probability p and tails with probability $q = 1 - p$.



We can actually solve this model explicitly (that's why we started with the simple model), but let's see how we would do things iteratively first. The key thing to notice is that you can only have n dollars at time $t+1$ if you had $n-1$ or $n+1$ dollars at time t. This is very diffrent from the last model, where it was possible for you to get unlucky and lose all your atoms almost immediately. In particular, if you know you have n atoms at time t, then $P_n(t) = 1$, and we have

$$P_{n+1}(t+1) = p \quad \text{and} \quad P_{n-1}(t+1) = q.$$

We usually don't know exactly how much money we have, but if we know the probabilities of each dollar amount, we can use the same idea to find the later probabilities. In particular,

$$\underbrace{P_n(t+1)}_{\substack{\text{probability of} \\ n \text{ dollars next} \\ \text{flip.}}} = \underbrace{p}_{\substack{\text{one more} \\ \text{head}}} \cdot \underbrace{P_{n-1}(t)}_{\substack{\text{one less dollar} \\ \text{last flip}}} + \underbrace{q}_{\substack{\text{one} \\ \text{tail}}} \cdot \underbrace{P_{n+1}(t)}_{\substack{\text{one more} \\ \text{dollar last} \\ \text{flip.}}}$$

This gives us an *infinite* system of coupled difference equations. Normally, this means that we have to resort to numerics to find all the $P_n(t)$, but in a few special cases we can find exact solutions.

To find our exact solution, we can go back to our diagram and notice the following: For a given n and t, there might be many ways to get to $N(t)=n$, but all possible paths have the same # of heads and tails.

In particular, suppose we want to have n dollars after t flips. Let h be the number of heads we got up to time t, so $t-h$ is the number of tails.

Our total winnings are therefore

$$+1 \cdot h - 1 \cdot (t-h) = 2h - t.$$

We need to win n dollars, so

$$2h - t = n \Rightarrow h = \frac{n+t}{2}.$$

(Note that this only makes sense if h is a whole number, so we need $n+t \geq 0$ and $n+t$ even, which means n and t are both even or both odd. Does this make sense from our diagram?)

It's easy to calculate the probability of any particular sequence of flips. For instance, the chance of x heads in a row followed by y tails is

$\Pr[x \text{ heads, then } y \text{ tails}] = p^x \cdot q^y = p^x \cdot (1-p)^y.$

If we want a total of x heads and y tails, we need to account for all the different orders these flips could come.

We know how to find the # of orders for x heads and y tails – we just choose x spots out of $x+y$ total, so there are

$$\binom{x+y}{x} = \frac{(x+y)!}{x!\,y!}$$

total ways.

Combining these results, we have

$$\Pr[N(t) = n] = \Pr\left[\tfrac{n+t}{2} \text{ heads, } \tfrac{t-n}{2} \text{ tails}\right]$$

$$= \left(\frac{t!}{\left(\frac{n+t}{2}\right)!\left(\frac{t-n}{2}\right)!}\right) p^{\left(\frac{n+t}{2}\right)} \cdot (1-p)^{\frac{t-n}{2}}.$$

This is almost the binomial distribution.

(we are assuming $\Pr[N(t)=n]=0$ if $n+t$ is not even or if $n+t<0$.)

To make it look like the binomial distribution, we need to change variables to get rid of this even/odd business. In particular, we will just look at even time steps, so $t = 2T$ and $n = 2x$. This means we have a new random variable $X(t) = \frac{N(t)}{2}$.

Substituting this into our last equation, we have

$$\Pr\left[N(t) = n\right] = \Pr\left[X(t) = x\right] = \frac{(2T)!}{(T+x)!\,(T-x)!}\, p^{T+x}\,(1-p)^{T-x}.$$

This still isn't quite there, so let $y = T + x = \frac{t}{2} + x$. This means $Y(t) = \frac{N(t)}{2} + \frac{t}{2}$.

Now we have

$$\Pr\left[N(t) = n\right] = \Pr\left[Y(t) = y\right] = \frac{(2T)!}{y!\,(2T-y)!}\, p^{y}\,(1-p)^{2T-y}.$$

Therefore,

$$\langle Y(t)\rangle = 2\tau p \quad \text{and} \quad \sigma(Y(t)) = 2\tau p(1-p).$$

Since we only made a linear change of variables, we can find $\langle N(t)\rangle$ and $\sigma(N(t))$ fairly easily:

$$\langle Y(t)\rangle = \frac{\langle N(t)\rangle}{2} + \frac{t}{2} = 2\tau p = tp$$

$$\Rightarrow \langle N(t)\rangle + t = 2tp$$

$$\Rightarrow \langle N(t)\rangle = 2tp - t$$

$$\Rightarrow \langle N(t)\rangle = t(2p-1).$$

The standard deviation is:

$$\sigma(Y(t)) = \frac{\sigma(N(t))}{2} = \sqrt{2\tau p(1-p)} = \sqrt{tp(1-p)}$$

$$\Rightarrow \sigma(N(t)) = 2\sqrt{tp(1-p)}$$