Overview:

Last class, we derived a stochastic model for population growth, called a linear birth-death process. In particular, if we let $N(t)$ = # of individuals at time t (a random variable) and define

$$P_n(t) = Pr[N(t) = n \mid N(0) = N_0],$$

then we obtained the system of ode's:

$$\begin{cases} \dfrac{dP_n}{dt} = \beta\cdot(n-1)\cdot P_{n-1} - (\beta+\delta)\cdot n\cdot P_n + \delta\cdot(n+1)\cdot P_{n+1} \\ \text{w/ } P_n(0) = \begin{cases} 1 & \text{if } n = N_0 \\ 0 & \text{if } n \neq N_0 \end{cases} \end{cases}$$

In your homework you solved a similar problem with $\delta = 0$ by finding P_0, then using that solution to find P_1, then using that to find P_2, etc. Unfortunately, that method will not work here, because each P_n depends on both P_{n-1} and P_{n+1}. (If you're familiar with linear algebra, this is the same issue as when you try to solve a tridiagonal system instead of a triangular one.)

The way to get around this issue is to somehow solve for all the P_n's at the same time. To do this, we will define the "probability generating function", which combines information about all the P_n's into one function. In particular,

$$F(t,x) = \sum_{n=0}^{\infty} P_n(t) x^n.$$

Whenever someone writes down an infinite series, the first thing you should do is decide if it converges. In our case, we know that $0 \le P_n(t) \le 1$ for all n and t, so

$$\sum_{n=0}^{\infty} 0 x^n \le \sum_{n=0}^{\infty} P_n(t) x^n \le \sum_{n=0}^{\infty} x^n.$$

The LHS is 0, so it converges for all x. The RHS is a geometric series, so it converges for all $|x|<1$. Therefore, $F(t,x)$ is well-defined for at least all $|x|<1$. Moreover, since the P_n's are probabilities, we have

$$F(t,1) = \sum_{n=0}^{\infty} P_n(t) \cdot 1^n = \sum_{n=0}^{\infty} P_n(t) = 1,$$

So the radius of convergence of $F(t,x)$ is at least $|x| \leq 1$. This means $F(t,x)$ is well-defined for all $|x| \leq 1$.

We now know that, given any probabilities $P_n(t)$, we can write the probability generating function. However, this process is only useful if we can also go backwards. That is, if we are given $F(t,x)$, we should be able to find each $P_n(t)$. The trick to notice is that $F(t,x)$ is essentially a taylor series. Remember that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$a_n = \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=0}.$$

In our case, the a_n's are actually functions of t, but that isn't a problem. We have

$$P_n(t) = \frac{1}{n!} \frac{\partial^n F}{\partial x^n}\bigg|_{x=0}.$$

This should make a lot of sense if we just write a few terms of our series.

For instance:

$$F(t,x) = P_0(t) + P_1(t)x + P_2(t)x^2 + \dots$$

So

$$\frac{1}{0!}\frac{\partial^0 F}{\partial x^0}\Big|_{x=0} = \frac{1}{1}F(t,0) = F(t,0) = P_0(t) + \cancel{P_1(t)\cdot 0}^{\,0} + \cancel{P_2(t)\,0^2}^{\,0} + \dots$$

And

$$\frac{1}{1!}\frac{\partial^1 F}{\partial x^1}\Big|_{x=0} = \frac{1}{1}\frac{\partial F}{\partial x}\Big|_{x=0} = 0 + P_1(t) + \cancel{2P_2(t)\cdot 0}^{\,0} + \dots = P_1(t)$$

And

$$\frac{1}{2!}\frac{\partial^2 F}{\partial x^2}\Big|_{x=0} = \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = \frac{1}{2}\left[0 + 0 + 2P_2(t) + \dots\right] = P_2(t)$$

Etc.

Even better, this function gives us an easy way to calculate useful statistics about N.

In particular,

$$\langle N(t)\rangle = \sum_{n=0}^{\infty} n P_n(t) = \sum_{n=0}^{\infty} n P_n(t)\cdot 1^{n-1} = \sum_{n=0}^{\infty} P_n(t)\cdot\left[n x^{n-1}\right]_{x=1}$$

$$= \frac{\partial F}{\partial x}\Big|_{x=1}.$$

Similarly,

$$\frac{\partial^2 F}{\partial x^2}\bigg|_{x=1} = \sum_{n=0}^{\infty} P_n(t)\cdot n\cdot(n-1)\cdot x^{n-2}\bigg|_{x=1}$$

$$= \sum_{n=0}^{\infty} (n^2 - n)\, P_n(t)$$

$$= \sum_{n=0}^{\infty} n^2 P_n(t) - \sum_{n=0}^{\infty} n\, P_n(t)$$

$$= \langle N(t)^2 \rangle - \langle N(t) \rangle.$$

Remember, the variance of N is

$$\mathrm{Var}[N(t)] = \langle N(t)^2 \rangle - \langle N(t) \rangle^2$$

$$= \underbrace{\langle N(t)^2 \rangle - \langle N(t) \rangle} + \langle N(t) \rangle - \langle N(t) \rangle^2$$

$$= \frac{\partial^2 F}{\partial x^2}\bigg|_{x=1} + \frac{\partial F}{\partial x}\bigg|_{x=1} - \left(\frac{\partial F}{\partial x}\bigg|_{x=1}\right)^2$$

$$= \left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial x} - \left(\frac{\partial F}{\partial x}\right)^2\right]_{x=1}$$

And therefore the standard deviation is

$$\sigma(N(t)) = \sqrt{\mathrm{Var}[N(t)]} = \sqrt{\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial x} - \left(\frac{\partial F}{\partial x}\right)^2\right]_{x=1}}$$

Similarly, we can calculate the coefficient of variation

$$C_v(N(t)) = \frac{\sigma(N(t))}{\langle N(t)\rangle}.$$

As you can see, once we have F, it is easy to find any information we want about p. Now the question becomes:

How do we find F?

Notice that

$$\frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} \frac{dP_n}{dt} x^n.$$

Using our system of ode's, this means that

$$\frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} \left[\beta\cdot(n-1)P_{n-1}(t) - (\beta+\delta)\, n\cdot P_n(t) + \delta\cdot(n+1)\cdot P_{n+1}(t)\right]x^n$$

$$= \beta\sum_{n=0}^{\infty}(n-1)P_{n-1}(t)x^n - (\beta+\delta)\sum_{n=0}^{\infty} nP_n(t)x^n + \delta\sum_{n=0}^{\infty}(n+1)P_{n+1}(t)x^n$$

Let's look at each of these sums in turn.

$$\sum_{n=0}^{\infty}(n-1)P_{n-1}(t)x^n = \sum_{n=1}^{\infty}(n-1)P_{n-1}(t)x^n \quad \left(\text{because } P_{-1}(t)=0\right)$$

If we let $k=n-1$, then

$$= \sum_{k=0}^{\infty} k P_k(t) x^{k+1}$$

$$= x^2 \sum_{k=0}^{\infty} k P_k(t) x^{k-1}$$

$$= x^2 \frac{\partial F}{\partial x}.$$

Likewise,

$$\sum_{n=0}^{\infty} n P_n(t) x^n = x \sum_{n=0}^{\infty} n P_n(t) x^{n-1} = x \frac{\partial F}{\partial x}$$

and

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(t)x^n = \sum_{k=1}^{\infty} k P_k(t) x^{k-1} \quad \leftarrow k=n+1$$

$$= \sum_{k=0}^{\infty} k P_k(t) x^{k-1} = \frac{\partial F}{\partial x}.$$

We therefore have

$$\frac{\partial F}{\partial t} = \beta x^2 \frac{\partial F}{\partial x} - (\beta + \delta) x \frac{\partial F}{\partial x} + \delta \frac{\partial F}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial t} = \left[\beta x^2 - (\beta+\delta)x + \delta\right] \frac{\partial F}{\partial x}}$$

In addition, since $P_n(0) = \begin{cases} 1 & n = N_0 \\ 0 & n \neq N_0 \end{cases}$, we know

that $\boxed{F(0,x) = x^{N_0}.}$

This is actually a Partial Differential Equation, rather than an ode. PDE's are typical of birth-death processes, since we always have to deal with an infinite dimensional system.

Most of you haven't had any experience with PDE's before. We won't use the general theory, but it turns out that this problem can be solved by turning it back into a (small) system of odes.

We will use a trick (really, this is a standard method called the "method of characteristics") to solve for F.

First, for notational convenience, let $t = \tau$.
Next, we will pretend that x is actually a function of τ rather than an independent variable.
That is, $x = u(\tau)$ for some unspecified function u.
Finally, let $v = u(0)$.

This means that F is really just a function of one variable, τ. We have

$F(t,x) = F(t(\tau), x(\tau))$, so (using the chain rule)

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial t}\frac{dt}{d\tau} + \frac{\partial F}{\partial x}\frac{dx}{d\tau}.$$

$$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{du}{d\tau}.$$

From our PDE, we have

$$\frac{\partial F}{\partial t} = \left[\beta x^2 - (\beta+\delta)x + \delta\right]\frac{\partial F}{\partial x}$$

$$\Rightarrow \frac{\partial F}{\partial t} - \left[\beta x^2 - (\beta+\delta)x + \delta\right]\frac{\partial F}{\partial x} = 0$$

So
$$\frac{\partial F}{\partial t} + (\beta x - \delta)(1-x)\frac{\partial F}{\partial x} = 0.$$

This means that if we choose u so that

$$\frac{du}{d\tau} = (\beta x - \delta)(1-x), \quad \text{then}$$
$$= (\beta u - \delta)(1-u)$$

$$\frac{\partial F}{\partial t} + \frac{du}{d\tau}\frac{\partial F}{\partial x} = 0,$$

so $$\frac{dF}{d\tau} = 0.$$

We therefore have a system of two ODE's:

$$\begin{cases} \dfrac{dF}{d\tau} = 0, \\ \dfrac{du}{d\tau} = (\beta u - \delta)(1-u), \end{cases}$$

with $\boxed{u(0) = V}$ and $\boxed{F(0) = F(0, u(0)) = F(0, V) = V^{N_0}}$

The first equation is easy to solve, and we get $F(\tau) = V^{N_0}$.

The problem is that we don't actually know what V is. To find it, we need to solve for u.

We have

$$\frac{du}{d\tau} = (\beta u - \delta)(1-u), \qquad u(0)=v.$$

This equation is separable, and therefore not too bad to solve. The method varies slightly if $\beta = \delta$ or if $\beta \neq \delta$. We will do this in two cases:

$\boxed{\beta = \delta}$:

$$\frac{du}{d\tau} = (\beta u - \delta)(1-u) = (\beta u - \beta)(1-u)$$
$$= -\beta \cdot (u-1)^2,$$

so $$\frac{-1}{(u-1)^2}\frac{du}{d\tau} = 1$$

so $$\int -(u-1)^{-2}\, du = \int 1 \, d\tau$$

$$\Rightarrow \quad \frac{1}{u-1} = \beta\tau + C.$$

Using our initial condition $u(0)=v$, we get

$$\frac{1}{u-1} = \beta\tau + \frac{1}{v-1},$$

$$\frac{1}{v-1} = \frac{1}{u-1} - \beta\tau = \frac{1-\beta\tau(u-1)}{u-1}$$

$$\Rightarrow v-1 = \frac{u-1}{1-\beta\tau u+\beta\tau}$$

$$\Rightarrow v = \frac{u-1}{1-\beta\tau u-\beta\tau} + 1 = \frac{u-1+1-\beta\tau u+\beta\tau}{1+\beta\tau-\beta\tau u}$$

so

$$v = \frac{\beta\tau + (1-\beta\tau)u}{1+\beta\tau-\beta\tau u}$$

Therefore,

$$F(\tau) = \left[\frac{\beta\tau + (1-\beta\tau)u}{1+\beta\tau-\beta\tau u}\right]^{N_0}.$$

But $\tau = t$ and $u = x$,

so

$$\boxed{F(t,x) = \left[\frac{\beta t + (1-\beta t)x}{1+\beta t-\beta t x}\right]^{N_0} \quad \text{when } \beta=\delta.}$$

If $\boxed{\beta \neq \delta}$:

$$\frac{du}{d\tau} = (\beta u - \delta)\cdot(1-u), \qquad u(0) = v$$

$$\Rightarrow \frac{1}{(1-u)(\beta u - \delta)} \frac{du}{d\tau} = 1,$$

Using partial fractions, we have

$$\frac{1}{(1-u)(\beta u-\delta)} = \frac{A}{1-u} + \frac{B}{\beta u - \delta}$$

$$\Rightarrow 1 = A(\beta u - \delta) + B(1-u) = (\beta A - B)u + (B - \delta A)$$

$$\Rightarrow \beta A - B = 0 \quad \text{and} \quad B - \delta A = 1$$

$$\Rightarrow B = \beta A \quad \Rightarrow \quad \beta A - \delta A = 1 \Rightarrow A = \frac{1}{\beta - \delta},$$

so $B = \dfrac{\beta}{\beta - \delta}$.

We therefore have

$$\frac{1}{\beta-\delta}\left[\frac{1}{1-u} + \frac{\beta}{\beta u - \delta}\right]\frac{du}{d\tau} = 1$$

So

$$\int \left[\frac{1}{1-u} + \frac{\beta}{\beta u - \delta}\right] du = \int (\beta - \delta)\, d\tau$$

$$\Rightarrow -\ln(|1-u|) + \ln(|\beta u - \delta|) = (\beta - \delta)\tau + C$$

$$\Rightarrow \ln\left(\left|\frac{\beta u - \delta}{1-u}\right|\right) = (\beta - \delta)\tau + C$$

$$\Rightarrow \left|\frac{\beta u - \delta}{1-u}\right| = Ce^{(\beta - \delta)\tau}.$$

Using the initial condition $u(0) = v$, we get

$$\frac{\beta u - \delta}{1-u} = \frac{\beta v - \delta}{1-v} e^{(\beta - \delta)\tau}.$$

This gives

$$(\beta u - \delta)(1-v) = (1-u)(\beta v - \delta)e^{rt} \quad (\text{where } r = \beta - \delta)$$

$$\Rightarrow \beta u - \delta - (\beta u - \delta)v = \beta e^{rt}(1-u)v - \delta e^{rt}(1-u)$$

$$\Rightarrow \beta e^{rt}(1-u)v + (\beta u - \delta)v = \beta u - \delta + \delta e^{rt}(1-u)$$

So we get

$$V = \frac{\delta e^{r\tau}(1-u) + (\beta u - \delta)}{\beta e^{r\tau}(1-u) + (\beta u - \delta)}$$

So

$$F(\tau) = \left[\frac{\delta e^{r\tau}(1-u) + (\beta u - \delta)}{\beta e^{r\tau}(1-u) + (\beta u - \delta)}\right]^{N_0}$$

and therefore

$$F(t,x) = \left[\frac{\delta e^{rt}(1-x) + (\beta x - \delta)}{\beta e^{rt}(1-x) + (\beta x - \delta)}\right]^{N_0}, \quad \beta \neq \delta$$