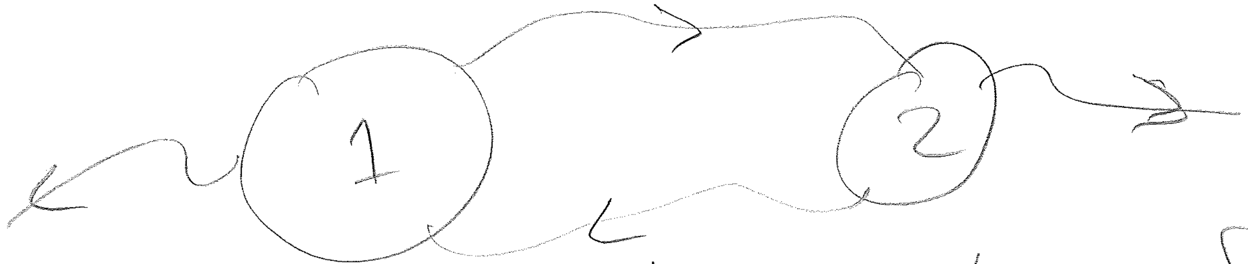
This lecture will mark a somewhat abrupt shift in topic. So far, we have talked about several different types of models: determenistic, stochastic, discrete time, continuous time, discrete space and continuous space. All of them have shared one common feature, though. They have all had one state variable. That is, we were only ever concerned with the population of one species or the value in one bank account. Some of these models (birth death processes and delay models, for example) needed more than equation, but our ultimate goal was to count one thing. We will now rectify this issue and look at systems with more than one state variable. Today, we will look at a very simple model in order to review some simple concepts.

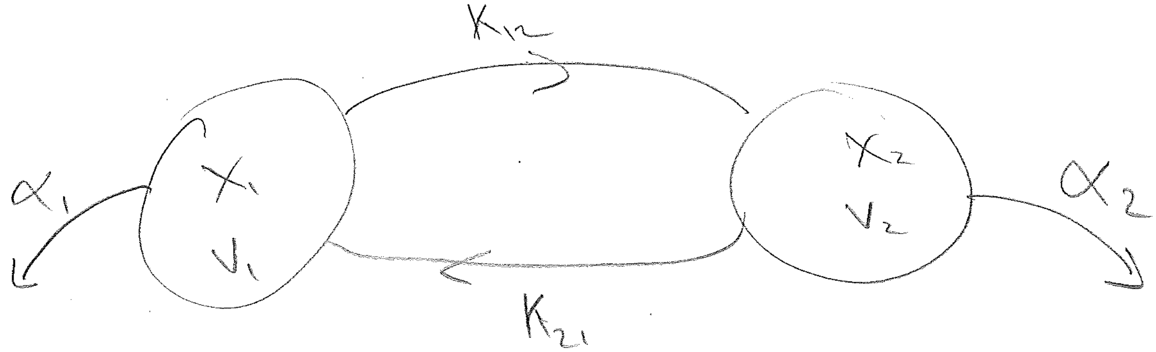
Suppose that we have two ponds connected by a network of streams.




We will imagine that some streams flow from pond $1 \to 2$ and others flow from $2 \to 1$. One day, a barrel of pollutant is dumped into pond 1. This pollutant will not be contained just to the first pond - it will spread through the streams to the other pond. We want to know how much pollutant there will be in each pond at any given time. We will assume that each pond circulates enough that the pollutant is well-mixed, so it makes sense to talk about the concentration of pollutant in each pond.

Let $X_1(t)$ and $X_2(t)$ represent the mass of pollutant in ponds 1 and 2, respectively, at time t. If $V_1$ and $V_2$ are the volumes of the two ponds, then $\frac{X_1(t)}{V_1}$ and $X_2(t)/V_2$ are concentrations.

In addition, let $K_{12}$ be the flow rate (in volume per time) from pond 1 to pond 2, let $K_{21}$ be the flow rate from pond 2 to pond 1, let $\alpha_1$ be the flow rate out of pond 1 and $\alpha_2$ be the flow rate out of pond 2.



We can now track the flows through each pond. Note that

$$\begin{bmatrix}\text{Rate of change}\\ \text{of } x_1\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{mass in}\end{bmatrix} - \begin{bmatrix}\text{rate of}\\ \text{mass out}\end{bmatrix}$$

$$= \begin{bmatrix}\text{Concentration}\\ \text{in Pond 2}\end{bmatrix} \cdot \begin{bmatrix}\text{flow rate}\end{bmatrix} - \begin{bmatrix}\text{concentration}\\ \text{in pond 1}\end{bmatrix} \cdot \begin{bmatrix}\text{flow}\\ \text{rate}\\ \text{out}\end{bmatrix}$$

so

$$\frac{dx_1}{dt} = \frac{x_2}{V_2} \cdot K_{21} - \frac{x_1}{V_1} \cdot (\alpha_1 + K_{12})$$

and likewise

$$\frac{dx_2}{dt} = \frac{x_1}{V_1} \cdot K_{12} - \frac{x_2}{V_2} \cdot (\alpha_2 + K_{21}).$$

This is a linear, constant coefficient system of equations. We usually write it as

$$\dot{\vec{x}} = \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} -\left(\frac{\alpha_1 + K_{12}}{V_1}\right)x_1 + \frac{K_{21}}{V_2}x_2 \\ \frac{K_{12}}{V_1}x_1 - \left(\frac{\alpha_2 + K_{21}}{V_2}\right)x_2 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha + K_{12}}{V_1} & \frac{K_{21}}{V_2} \\ \frac{K_{12}}{V_1} & -\frac{\alpha_2 + K_{21}}{V_2} \end{pmatrix} \cdot \begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

This is an example of a much more general class of systems that you should all be familiar with already:

$\dot{\vec{x}} = A\vec{x}$, where $\vec{x}$ is an $n \times 1$ vector of unknown functions and $A$ is an $n \times n$ constant matrix. We will quickly review the solution to these equations.

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The simplest such system is when $n=1$, so $A$ is just one number. In this case,

$\dot{x}_1 = \lambda x_1$ and the solution is

$x_1(t) = Ce^{\lambda t}$.

By analogy, we will guess that each $x_k$ is of the form $x_k(t) = C_k e^{\lambda t}$, where the $C_k$'s may vary with $k$, but $\lambda$ does not.

This means $\vec{x}(t) = \vec{v} \cdot e^{\lambda t}$, where $\vec{v}$ is a constant $n \times 1$ vector.

Substituting this into our ode, we get

$$\frac{d}{dt}(\vec{v}e^{\lambda t}) = A(\vec{v}e^{\lambda t}), \text{ so}$$

$$\lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t} \text{ and therefore}$$

$$\lambda\vec{v} = A\vec{v}, \text{ so}$$

$$A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I_n\vec{v} = 0, \text{ where } I_n$$

is the $n \times n$ identity matrix. This means

$$(A - \lambda I_n)\vec{v} = \vec{0}.$$

One option, of course, is that $\vec{v} = \vec{0}$ and $\lambda$ is arbitrary, but this corresponds to the useless zero solution. If there are any other solutions for $\vec{v}$, then $A - \lambda I_n$ must be singular, so $\det(A - \lambda I_n) = 0$.

This means that $\vec{v}$ is an eigenvector of $A$ and $\lambda$ is the associated eigenvalue.

Examples:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 0 & -1-\lambda \end{bmatrix}$$

so $\det(A - \lambda I_2) = (1-\lambda)(-1-\lambda) - 2\cdot 0 = 0$

$\Rightarrow (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \lambda = 1$ or $\lambda = -1$.

This is fairly typical. If $A$ is an $n \times n$ matrix, we get $n$ eigenvalues.

For $\lambda = 1$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 1-1 & 2 \\ 0 & -1-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0v_1 + 2v_2 = 0 \\ 0v_1 - 2v_2 = 0 \end{cases}$$

$\Rightarrow v_2 = 0$ and $v_1$ is arbitrary.

This means $\vec{v} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $c$ is an arbitrary constant.

Likewise, for $\lambda = -1$:

$$(A-\lambda I)\vec{v} = \begin{pmatrix} 1+1 & 2 \\ 0 & -1+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2v_1 + 2v_2 = 0 \\ 0v_1 + 0v_2 = 0 \end{cases} \Rightarrow v_1 = -v_2.$$

This means

$$\vec{v} = c \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ where $c$ is arbitrary.

We found two solutions:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \quad \text{and} \quad \vec{x}(t) = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

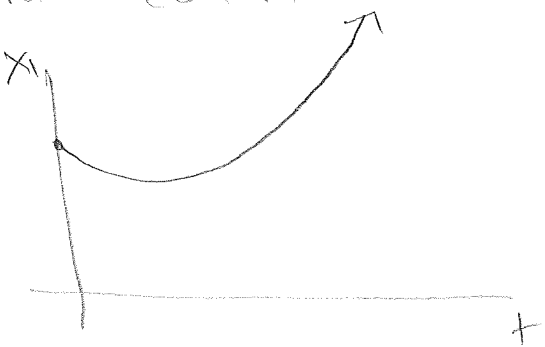
The full solution will be a linear combination of these, so

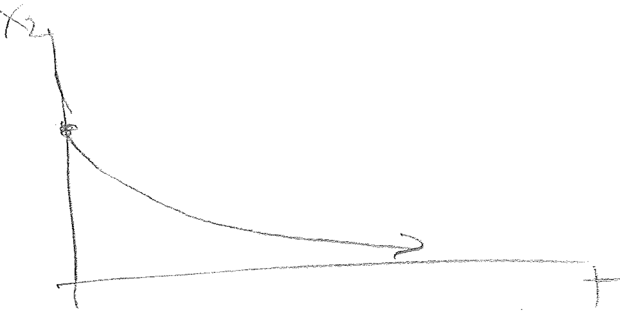
$$\vec{x}(t) = c_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

$$\Rightarrow \begin{cases} x_1(t) = c_1 e^{t} + c_2 e^{-t} \\ x_2(t) = -c_2 e^{-t} \end{cases}$$

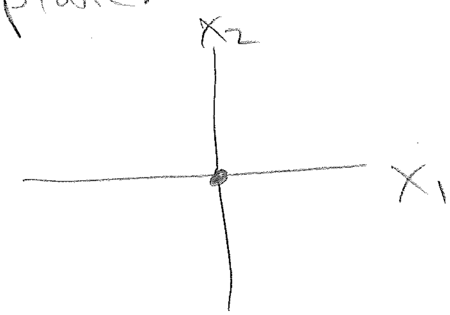
How can we visualize these solutions?

One way is to graph $x_1$ and $x_2$ vs $t$ for various initial conditions:

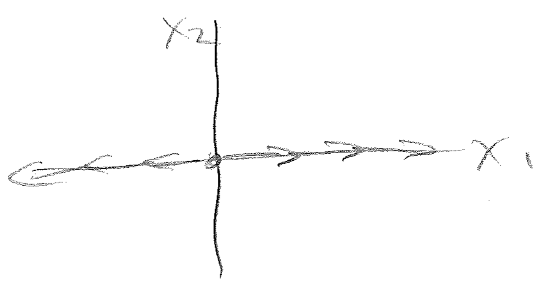



A more informative method is to graph $x_1$ vs $x_2$. This is called a "phase plane". We will slowly fill in this graph. First, notice that if $C_1 = C_2 = 0$, then $x_1(t) = x_2(t) = 0$ forever. This solution will just stay at the origin of the phase plane.




If $C_2 = 0$ but $C_1 \neq 0$, then $x_2(t) = 0$ forever, but $x_2(t)$ grows exponentially. This means that solutions stay on the $x_1$-axis but move away from the origin.

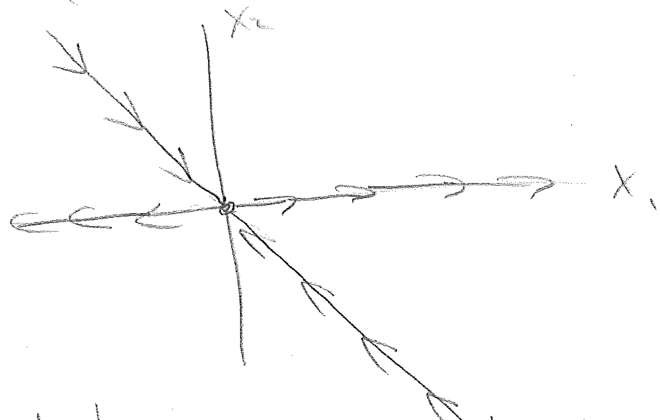


Likewise, if $C_1 = 0$ but $C_2 \neq 0$, then

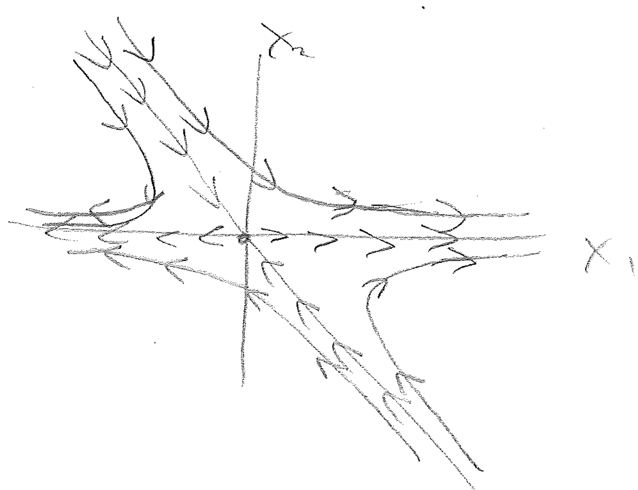
$$x_1(t) = C_2 e^{-t} \text{ and } x_2(t) = -C_2 e^{-t},$$

so $x_1(t) = -x_2(t)$ forever.

This means that the solutions stay on the line $x_1 = -x_2$ (and move towards zero).




Other solutions are harder to see directly, but we know that small changes in C don't change the graph much, so we can fill in nearby curves graphically:




There's an easy way to get this picture. The eigenvectors are $\binom{1}{0}$ and $\binom{1}{-1}$, so we draw a line between the origin and $(1,0)$ and another between the origin and $(1,-1)$.

If the corresponding eigenvalue is positive, we draw outward arrows on that line. If it is negative, we draw inward arrows.