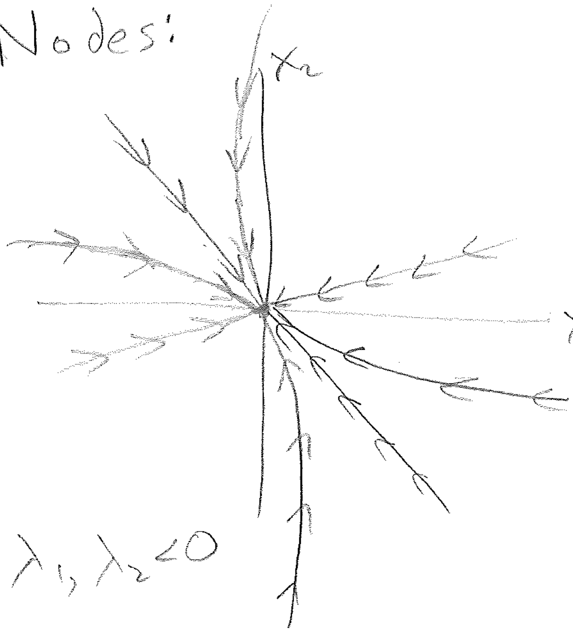
We are solving linear systems of ode's.

Recall that, for 2x2 systems, we have three different qualitative types of solutions:

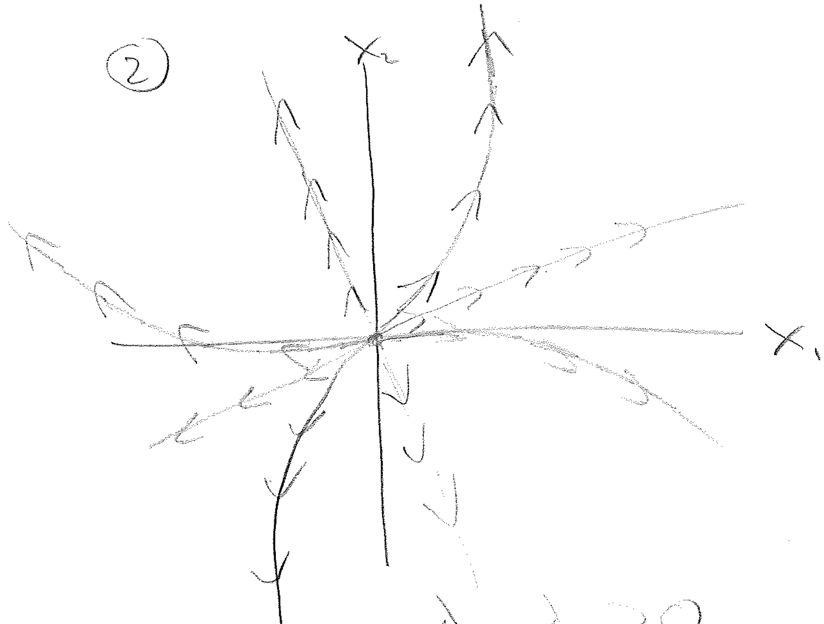
Nodes:

①



$\lambda_1, \lambda_2 < 0$

②

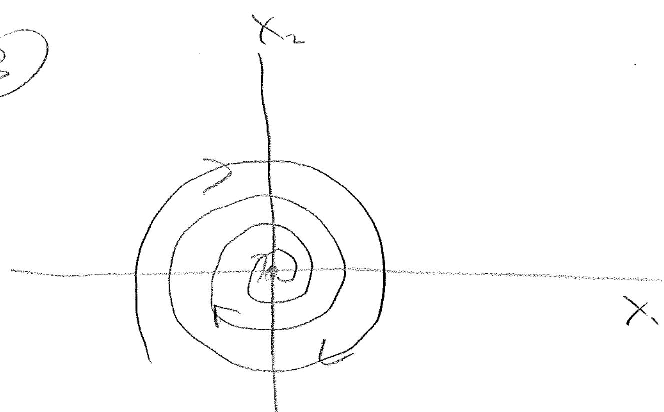


$\lambda_1, \lambda_2 > 0$

Two real eigenvalues, same sign

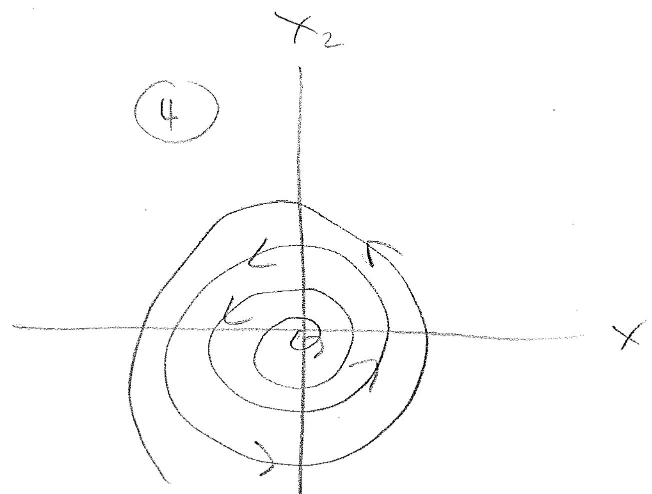
Spirals:

③



$a < 0$

④



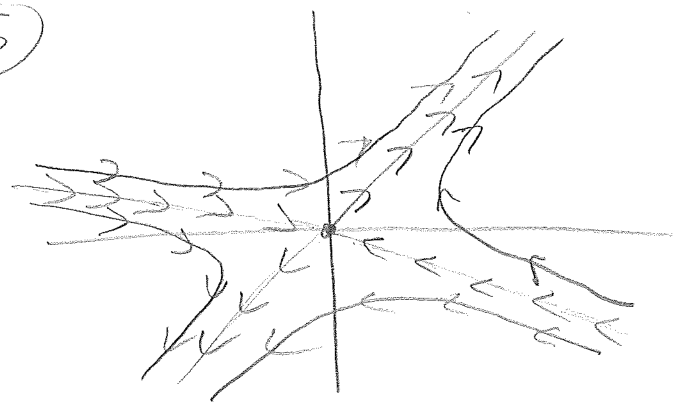
$a > 0$

Two complex eigenvalues

$\lambda = a \pm ib$

Saddle:

5



Two real eigenvalues, Different signs.

Unfortunately, these categories don't generalize all that well to higher dimensions. A 4-dimensional system, for instance, could have four complex eigenvalues with all the same sign of Real part, or two w/ positive and two w/ negative, or two complex eigenvalues and two real, or all four real with any combination of signs. For the most part, we don't bother naming all of these types. Instead, we usually categorize these systems by a familiar concept - stability.

Roughly speaking, we want to say that the equilibrium of these systems (always at the origin) is stable if solutions that start close to the origin get closer and unstable if solutions that start close to the origin move away.

This gives us a very useful rule:

If $\dot{\vec{x}} = A\vec{x}$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A, then the origin is a stable equilibrium if $Re(\lambda_k) < 0$ for <u>all</u> λ_k. The origin is unstable if $Re(\lambda_k) > 0$ for <u>any</u> λ_k.

Proof: If all the eigenvalues are real and negative, then

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + \dots + c_n \vec{v}_n e^{\lambda_n t},$$

but each term goes to zero as $t \to \infty$.

If there is some $\lambda_k > 0$, then one possible solution is

$$\vec{x}(t) = c_k \vec{v}_k e^{\lambda_k t},$$

which goes to ∞ as $t \to \infty$.

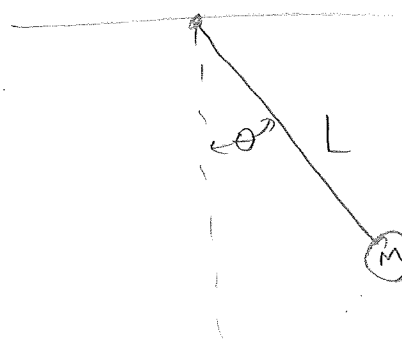
If the eigenvalues are complex, the same argument works, but when $\lambda_k = a_k + i b_k$,

$$e^{\lambda_k t} = e^{a_k t} \cdot \left(\cos(b_k t) + i \sin(b_k t) \right),$$

which only goes to zero if $a_k < 0$.

Example:

Suppose we have a pendulum hanging on a light, rigid rod.



For small angles of displacement (i.e. $\theta \ll 1$), we can model this system by

$$ML\ddot{\theta} = -\gamma \cdot \dot{\theta} - Mg\,\theta,$$

where M is the mass of the bob, g is the acceleration of gravity and γ is the coefficient of friction and L is the length.

This gives

$$\ddot{\theta} = -a\dot{\theta} - b\theta,$$

where $a = \frac{\gamma}{ML}$ and $b = \frac{g}{L}$.

If we let $x_1 = \theta$ and $x_2 = \dot{\theta}$, then

$\dot{x}_1 = \dot{\theta} = x_2$ and $\dot{x}_2 = \ddot{\theta} = -a\dot{\theta} - b\theta = -ax_2 - bx_1$,

so
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This is a linear system of ode's. It has a trivial (equilibrium) solution at $x_1 = x_2 = 0$. This corresponds to a pendulum hanging straight down with no velocity. Is this solution stable? What happens if the bob is moved a little? We can find out by finding the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

We have $A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{pmatrix}$, so

$$\det(A - \lambda I) = -\lambda(-a-\lambda) + b$$
$$= \lambda^2 + a\lambda + b = 0$$

$$\Rightarrow \lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If $a^2 - 4b > 0$, then both eigenvalues are real. Since $a, b > 0$, $\sqrt{a^2-4b} < a$, so both eigenvalues would then be negative.

This situation corresponds to a stable node. In particular, the solution does not oscillate. If $a^2 > 4b \Rightarrow a > 2\sqrt{b}$, then the force of friction is much stronger than that of gravity, so the pendulum drops to the bottom and stops.

On the other hand, if $a^2 - 4b < 0$ (so $a < 2\sqrt{b}$) then both eigenvalues are complex and $Re(\lambda) = -\frac{a}{2} < 0$, so the equilibrium is a stable spiral. This means that when friction isn't too strong, the pendulum swings back and forth as ~~it~~ slows down.

To complete our qualitative analysis, we should draw a phase diagram for each case.

1) $a^2-4b>0$

$$\Rightarrow \lambda = \frac{-a \pm \sqrt{a^2-4b}}{2} < 0.$$

$\vec{v}$ is an eigenvector if

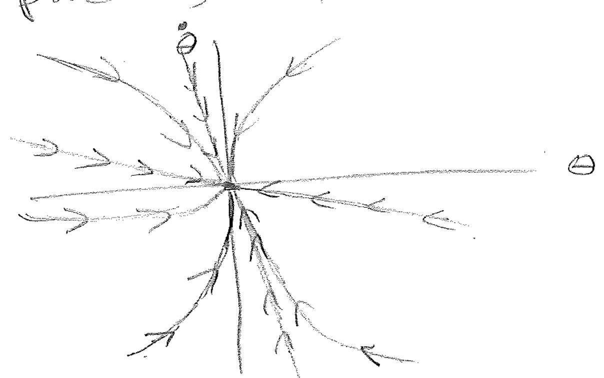
$$(A-\lambda I)\vec{v} = \begin{pmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This looks messy, but remember that this equation must be degenerate. That means that if we find a solution to the first equation it will also be a solution to the second. This makes it very easy to check that

$$\vec{v} = c \cdot \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \quad \text{so}$$

$$\vec{v}_1 = c_1 \cdot \begin{pmatrix} 1 \\ \frac{-a+\sqrt{a^2-4b}}{2} \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = c_2 \cdot \begin{pmatrix} 1 \\ \frac{-a-\sqrt{a^2-4b}}{2} \end{pmatrix}.$$

Our phase plane is therefore



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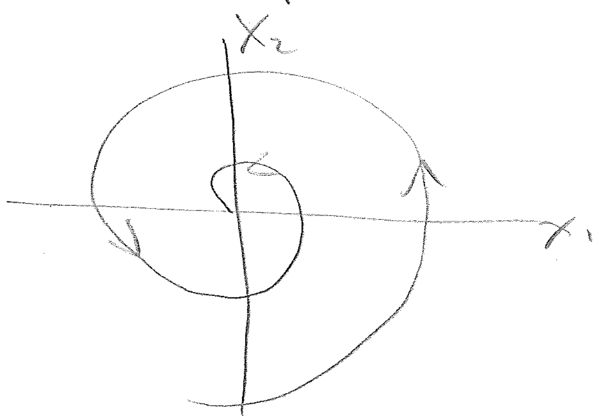
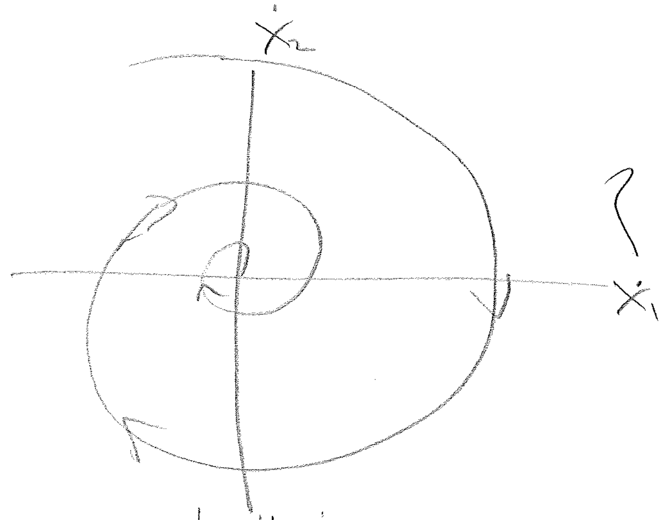
2) If $a^2 - 4b > 0$, we don't really need the eigenvectors to draw our picture, since we have a spiral. We should, however, check if the spiral turns clockwise or counterclockwise.



counter clockwise

or



Clockwise

To determine this, notice that when $x_1 > 0$ and $x_2 = 0$, x_2 is increasing for the ccw graph and x_2 is decreasing for the cw graph. In our case, if $x_1 > 0$ and $x_2 = 0$, we have $\dot{x}_2 = -bx_1 - ax_2$ $= -bx_1 < 0$, so we must have the clockwise graph.

Unfortunately, most real-world systems are not actually linear. Even our pendulum example is something of a hack. The actual equation of motion is not

$$\ddot{\theta} = -a\dot{\theta} - b\theta,$$

it is really

$$\ddot{\theta} = -a\dot{\theta} - b\sin\theta.$$

We can still substitute $x_1 = \theta$ and $x_2 = \dot{\theta}$ but now we get

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -b\sin(x_1) - ax_2. \end{cases}$$

This is decidedly nonlinear. How can we analyze it?

Our approach will be analogous to the method we employed for single nonlinear ode's: We will find equilibria, then look at linear approximations to the system near these equilibria.