Compound Interest

Bank 1: 6% APR (annual percentage rate)
6.17% APY (" " yield)

Bank 2: 6% APR
6.18% APY

Why the difference?

6% APR means that at the end of the year you get a 6% increase in \$ so

$$P(t+1) = (1+0.06) \cdot P(t)$$

APY is different because income is compounded more often

E.g. monthly

$$P(t+1) = \left(1+\frac{0.06}{12}\right)^{12} P(t) \approx 0.0617$$

or daily

$$P(t+1) = \left(1+\frac{0.06}{52}\right)^{52} P(t) \approx 0.0618$$

Compounding interest more often gives a higher yield.

In general if interest compounded every Δt years, then

$$P(t+\Delta t) = (1+.06\,\Delta t)\,P(t)$$

$$\Rightarrow P(t) = (1+.06\,\Delta t)^{t/\Delta t}\,P(0)$$

What if we compound really often?

$$P(t+\Delta t) = \left(1 + r\Delta t\right) P(t)$$

$$\Rightarrow P(t+\Delta t) = P(t) + r\Delta t\, P(t)$$

$$\Rightarrow \frac{P(t+\Delta t) - P(t)}{\Delta t} = r P(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = r P(t)$$

$$\Rightarrow \frac{d}{dt} P(t) = r P(t)$$

$$\Rightarrow P(t) = e^{rt} \cdot P(0)$$

Other way:

$$P(t) = \left(1 + r\Delta t\right)^{t/\Delta t} P(0)$$

$$= \left(\left(1 + r\Delta t\right)^{\frac{1}{\Delta t}}\right)^{t} \cdot P(0)$$

so

$$\lim_{\Delta t \to 0} P(t) = \lim_{\Delta t \to 0} \left(\left(1 + r\Delta t\right)^{\frac{1}{\Delta t}}\right)^{t} \cdot P(0)$$

$$= \left(e^{r}\right)^{t} \cdot P(0)$$

$$= e^{rt} \cdot P(0)$$

Lemma:

$$\lim_{x \to 0} \left(1 + rx\right)^{1/x} = L$$

$$\lim_{x \to 0} \frac{1}{x} \ln(1 + rx) = \ln L$$

$$= \lim_{x \to 0} \frac{\ln(1 + rx)}{x} \quad \text{L'Hopital's Rule}$$

$$= \lim_{x \to 0} \frac{\frac{1}{1+rx} \cdot r}{1} = r$$

$$\Rightarrow \ln L = r$$

$$\Rightarrow L = e^{r}$$

Mortgage Payments

You borrow $P(0)$ dollars. Every Δt years, you make accrue interest and make a payment

usually $1/12$ or $1/24$

rate r — M dollars

We have:

$$P(t+\Delta t) = P(t)\cdot \underbrace{(1+r\Delta t)}_{R} - M$$

$$\Rightarrow P(t+\Delta t) = RP(t) - M$$

$$P(\Delta t) = RP_0 - M$$

$$P(2\Delta t) = R[RP_0 - M] - M = R^2P_0 - RM - M$$

$$P(3\Delta t) = R[R^2P_0 - RM - M] - M$$

$$= R^3P_0 - R^2M - RM - M$$

$$P(4\Delta t) = R[R^3P_0 - R^2M - RM - M] - M$$

$$= R^4P_0 - R^3M - R^2M - RM - M$$

$$\vdots$$

$$P(n\Delta t) = R^nP_0 - (1 + R + R^2 + \dots + R^{n-1})M$$

Geometric Series

$$S = 1 + R + R^2 + \dots + R^{n-1}$$

$$RS = R + R^2 + \dots + R^n$$

$$S - RS = 1 - R^n$$

$$S(1-R) = 1 - R^n$$

$$S = \frac{1-R^n}{1-R}$$

$$P(n\Delta t) = R^nP_0 - \frac{1-R^n}{1-R}M$$

$$\Rightarrow P(t) = R^{t/\Delta t}P_0 - \frac{1-R^{t/\Delta t}}{1-R}M$$

If you want to pay off your loan in T years, we need $P(T) = 0$,

so

$$R^{T/\Delta t} P_0 - \frac{1 - R^{T/\Delta t}}{1 - R} M = 0$$

$$\Rightarrow R^{T/\Delta t} P_0 = \frac{1 - R^{T/\Delta t}}{1 - R} M$$

$$\Rightarrow \frac{(1-R) R^{T/\Delta t} P_0}{1 - R^{T/\Delta t}} = M$$

$$\Rightarrow \frac{-r\Delta t (1 + r\Delta t)^{T/\Delta t} P_0}{1 - (1 + r\Delta t)^{T/\Delta t}} = M$$

$$\Rightarrow M = \frac{r\Delta t (1 + r\Delta t)^{T/\Delta t} P_0}{(1 + r\Delta t)^{T/\Delta t} - 1} = P_0 \, r\Delta t \cdot \left(\frac{(1 + r\Delta t)^{T/\Delta t}}{(1 + r\Delta t)^{T/\Delta t} - 1} \right)$$

$$\left(\text{total payment} = M \cdot \frac{T}{\Delta t} \to \frac{e^{rT}}{e^{rT} - 1} \cdot rT \cdot P_0 \right)$$

Actually most mortgages are compounded more often than they are paid off, so interest compounds every τ, but payments are made every Δt.

Finding $P(t)$ for arbitrary multiples of τ is now annoying, but it's not too bad to find $P(n\Delta t)$.

In between payments, we have

$$P(t + \tau) = (1 + r\tau)\, P(t)$$

so $$P(t + m\tau) = (1 + r\tau)^{m} P(t)$$

It takes $\Delta t / \tau$ steps between payments, so

$$P(t + \Delta t) = (1 + r\tau)^{\Delta t/\tau} P(t) - M$$

This is the same as before, except $R = (1 + r\tau)^{\Delta t/\tau}$,

so $$P(t) = R^{t/\Delta t} P_0 - \frac{1 - R^{t/\Delta t}}{1 - R} M$$

$$= (1 + r\tau)^{t/\tau} P_0 + \frac{1 - (1 + r\tau)^{t/\tau}}{1\ r\tau} M$$

Pay off your loan in T years with

τ-compounded interest

$$P(T) = R^{T/\Delta t} P_0 - \left(\frac{1 - R^{T/\Delta t}}{1 - R} \right) M$$

$$\Rightarrow \quad M = \frac{(1-R)\, R^{T/\Delta t} P_0}{1 - R^{T/\Delta t}}$$

$$= \frac{\left(1 - (1 + r\tau)^{\Delta t/\tau}\right) (1 + r\tau)^{T/\tau} P_0}{1 - (1 + r\tau)^{T/\tau}}$$

$$\lim_{\tau \to 0} M = \frac{e^{rT} \left(e^{r\Delta t} - 1\right)}{e^{rT} - 1} \cdot P_0$$