Overview:

What models have we done so far?

- Falling objects $h'' = g$
- Population growth/ Compound interest

$$N(t+\Delta t) = (1+r\Delta t)N(t)$$

$$\Rightarrow \frac{dN}{dt} = rN$$

- Growth + constant harvesting / Mortgage payments

$$P(t+\Delta t) = (1+r\Delta t)P(t) - M(\Delta t)$$

$$\Rightarrow \frac{dP}{dt} = rP(t) - M \quad \text{(for example)}$$

These are, in many ways, the simplest models we can come up with. They're 1st order, they all have analytical solutions, they involve only constant and/or linear terms, they're deterministic.

These models (especially (2)) will form the basis for everything else we do - everything else is just adding in complications/realism.

This week, we'll add our first wrinkle: nonlinearity.

Since ODEs are generally easier to work with than difference equations, we'll start with continuous time models.

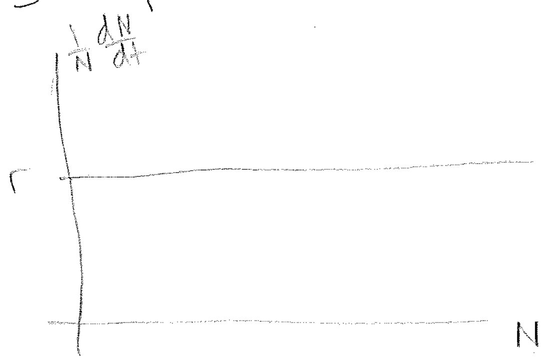
Back to Population growth:

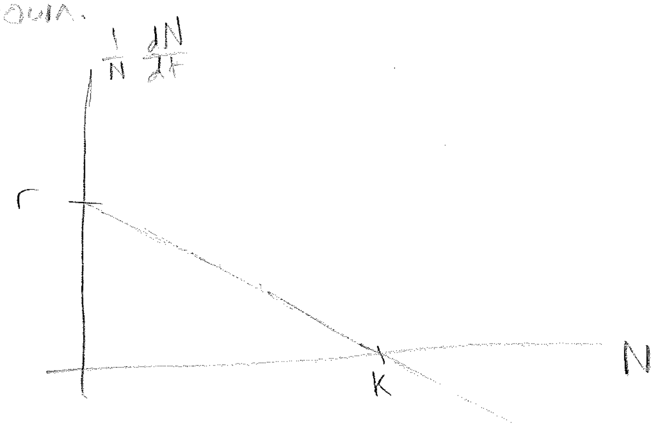
$$\frac{dN}{dt} = rN, \quad \text{where } N \text{ is the population and } r \text{ is the growth rate per organism}$$

In particular, notice that

$$r = \frac{1}{N}\frac{dN}{dt} = \text{the per capita growth rate.}$$

That is, the per capita growth rate is <u>constant</u> w/ respect to both time and population. This doesn't really make sense: if it's more crowded, mortality should go up, so r should go down.





The simplest way to add this density dependence is to make $\frac{1}{N}\frac{dN}{dt} = R(N)$ decay linearly, w/ $R(0) = r$ and $R = 0$ at some large population K. I.e.,

$$R(N) = r\cdot\left(1 - \frac{N}{K}\right), \quad \text{so}$$

$$\frac{1}{N}\frac{dN}{dt} = r\cdot\left(1 - \frac{N}{K}\right) \Rightarrow \frac{dN}{dt} = r\cdot N\cdot\left(1 - \frac{N}{K}\right).$$

This is called the logistic equation (or occasionally the Pearl-Verhulst equation)

Logistic Growth:

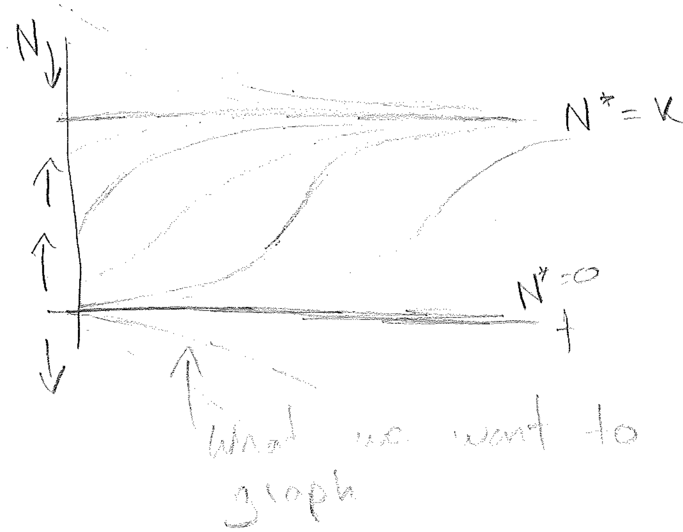
We can actually solve this equation explicitly, but we won't always be so lucky. With that in mind, let's see what we can understand without finding $N(t)$.

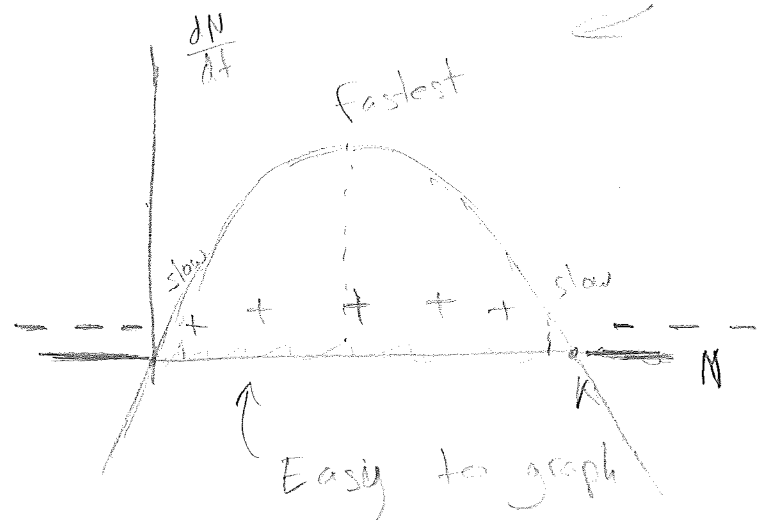
First, we want to find out when the population doesn't change. That is, we're looking for values of N where $\frac{dN}{dt} = 0$.

$$\frac{dN}{dt} = r \cdot N \cdot \left(1 - \frac{N}{K}\right) = 0 \Rightarrow N = 0 \quad \text{or} \quad 1 - \frac{N}{K} = 0 \Rightarrow N = K$$

Two equilibria: $N^* = 0$ and $N^* = K$

assume $r > 0$





$N^* = 0$ is called <u>unstable</u> – solutions move away from it.

$N^* = K$ is called <u>stable</u> – Solutions move towards it.

Logistic Growth continued:

Stability looks obvious from the graph, but we shouldn't trust graphs too much. Can we prove it?

Before we do any analysis we should make a change of variables. Stability involves the idea that N is "close to" N^*, which means $N - N^*$ is small, but we said variables with units shouldn't be called small or large. Instead, we should compare them to some other variable w/ the same units. To this end, let $x = \frac{N}{K}$. (unitless)

$$\Rightarrow N = Kx, \text{ so } \frac{dN}{dt} = K\frac{dx}{dt} \Rightarrow$$

$$\frac{dN}{dt} = r \cdot N \cdot \left(1 - \frac{N}{K}\right)$$

$$\downarrow$$

$$\cancel{K}\frac{dx}{dt} = r \cdot \cancel{K}x \cdot (1 - x)$$

$$\Rightarrow \dot{x} = rx \cdot (1 - x)$$

This means $x^* = 0$ and $x^* = 1$

Stability of $x^* = 0$:

Suppose x is close to $x^* = 0$ – i.e., $|x| \ll 1$.

$\frac{dx}{dt} = rx - rx^2$. If x is small, then x^2 is much smaller, so

$\frac{dx}{dt} \approx rx$ $\rightarrow$ This is just exponential growth. Since $r > 0$, solutions move away from 0, so $x^* = 0$ is unstable.

Stability of $x^* = 1$.

Suppose x is close to 1. That is, $x-1$ is small.
To make life easier, let $y = x-1$, so $y \ll 1$.

$$\frac{dy}{dt} = \frac{dx}{dt}$$

We have $\frac{dy}{dt} = r\cdot(y+1)\cdot(1-(y+1)) = r\cdot(y+1)\cdot(-y) = -ry^2 - ry$.

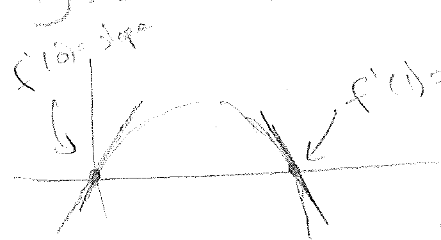
Since y is small, y^2 is much smaller, so

$\frac{dy}{dt} \approx -ry$. This is exponential decay. Since $r>0$, solutions (to y) move towards 0, so $y \to 0$, so $x \to 1$, so $x^* = 1$ is stable.

Notice: $\dot{x} = r\cdot x\cdot(1-x) = f(x)$

$f'(x) = r - 2rx$, so $f'(0) = r$ and $f'(1) = -r$.

These are the same as the coefficients we found in our stability analysis. Is that a coincidence?



If slope of tangent line is positive, then $\frac{dx}{dt} > 0$ for $x > x^*$ and $\frac{dx}{dt} < 0$ for $x < x^*$. If slope is negative, $\frac{dx}{dt} < 0$ for $x > x^*$ and $\frac{dx}{dt} > 0$ for $x < x^*$.

In general, x^* is stable if $f'(x^*) < 0$ and unstable if $f'(x^*) > 0$.

A slightly more interesting example (but a bad population model):

$$\frac{dx}{dt} = rx(1-x) - h, \quad \text{where } h \text{ is a constant (and } r > 0).$$

What are the equilibria and are they stable?

$$rx(1-x) - h = 0$$

$$\Rightarrow rx - rx^2 - h = 0$$

$$\Rightarrow rx^2 - rx + h = 0$$

$$\Rightarrow x = \frac{r \pm \sqrt{r^2 - 4rh}}{2r}$$

# of fixed points depends on r and h.

If $r^2 - 4rh > 0 \Rightarrow r - 4h > 0 \Rightarrow h < \frac{r}{4}$,

then there are two fixed points.

$$x_1^* = \frac{r + \sqrt{r^2 - 4h}}{2r}, \quad x_2^* = \frac{r - \sqrt{r^2 - 4h}}{2r}$$

Are they stable?

$f'(x) = r - 2rx$, so

$$f'(x_1^*) = r - 2r\cdot\left(\frac{r + \sqrt{r^2 - 4h}}{2r}\right)$$
$$= r - r - \sqrt{r^2 - 4h}$$
$$= -\sqrt{r^2 - 4h} < 0,$$

so x_1^* is stable

$$f'(x_2^*) = r - 2r\cdot\left(\frac{r - \sqrt{r^2 - 4h}}{2r}\right)$$
$$= r - r + \sqrt{r^2 - 4h}$$
$$= \sqrt{r^2 - 4h} > 0,$$

so x_2^* is unstable

If $r^2 - 4h = 0$, so $h = \frac{r}{4}$, there is only one equilibrium.

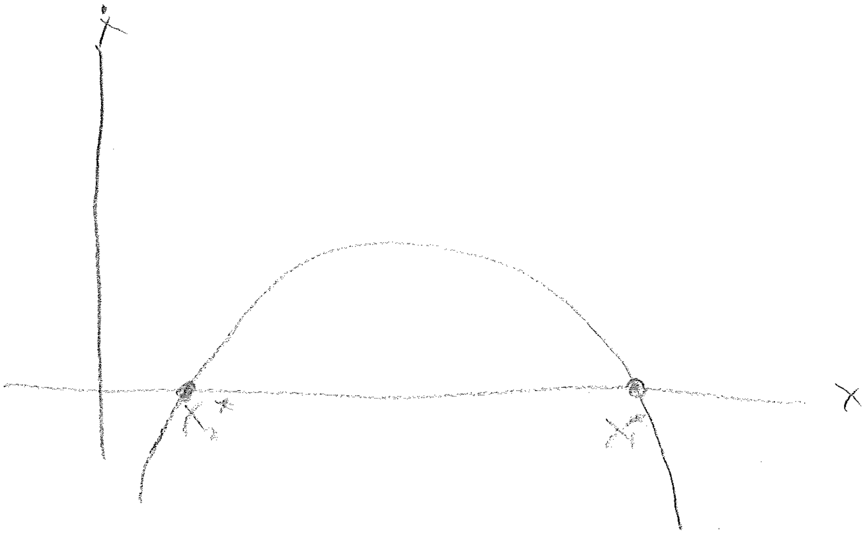
$$x^* = \frac{r}{2r} = \frac{1}{2}.$$

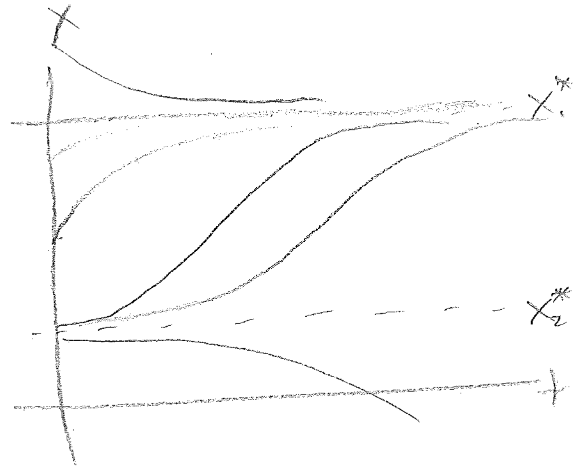
$$f'(x^*) = r - 2r\cdot\left(\frac{1}{2}\right) = r - r = 0\ ?$$

If $r^2 - 4h < 0$, there are no equilibria.

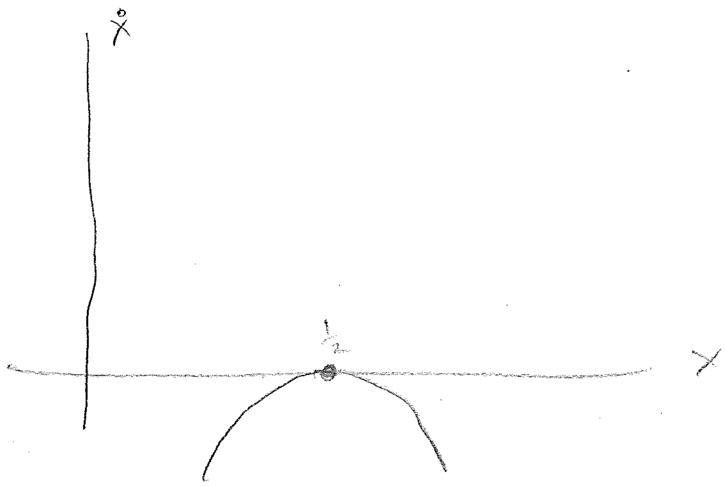
Phase line:

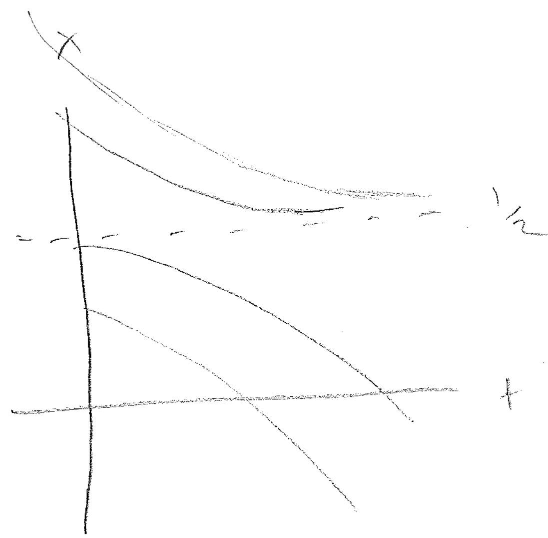
$h > \frac{r}{4}$





$h = \frac{r}{4}$





$h < \frac{r}{4}$