Overview:

Last time we talked about density-dependent (nonlinear) models for population growth. In particular, we investigated

$$\dot{N} = r \cdot N \cdot \left(1 - \frac{N}{K}\right) \quad - \text{Logistic growth}$$

We also talked briefly about a model w/ constant harvesting

$$\dot{x} = r x (1 - x) - h$$

Today we'll look at harvesting models in more detail, ...

Suppose you are in charge of the Fish and Wildlife department for some region, and you need to decide how to regulate fishing. Two options have been proposed:

- Selling permits by total # of fish caught
  e.g., a fisherman can buy a permit to catch 10 fish, and then once he catches 10 he has to leave
- Selling permits by total time spent fishing
  e.g. a fisherman can buy a permit for one day. He fishes for one day, keeps everything he catches and goes home

Fishery Model:

You already know that the fish population grows logistically w/ growth rate r and carrying capacity K. Our model should therefore be

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - H(N)$$

where N = # of fish and $H(N)$ is a function describing how many fish are caught per year (depending on population)

How do we choose H? Option 1: Sell constant H permits per year. Assuming the fishermen are very good and/or have a lot of spare time, we can assume that they will catch H fish per year, so $H(N)=H$ is constant

Therefore, $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - H$.

If we let $x = \frac{N}{K}$, so $Kx = N$ and $K\dot{x} = \dot{N}$ and let $h = \frac{H}{K}$,

then $K\dot{x} = rKx(1-x) - H$

$\Rightarrow \dot{x} = rx(1-x) - \frac{H}{K}$

$\Rightarrow \dot{x} = rx(1-x) - h$

Constant rate harvest:

We already described the solutions to this model.

If $h < \frac{r}{4}$, we have two equilibria:

$$\underbrace{x_1^* = \frac{r - \sqrt{r^2 - 4rh}}{2r}}_{\text{unstable}} \qquad \underbrace{x_2^* = \frac{r + \sqrt{r^2 - 4rh}}{2r}}_{\text{Stable}}$$

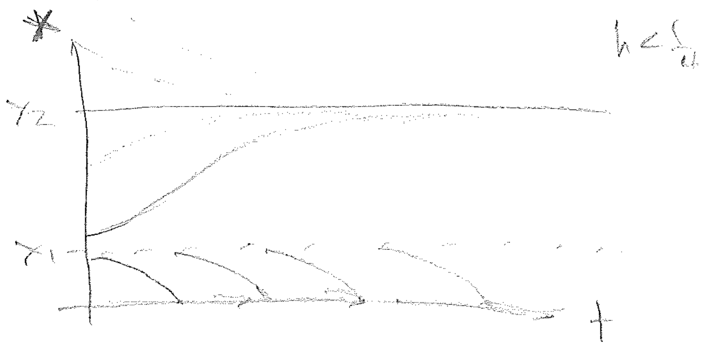
If $h = \frac{r}{4}$, we have one equilibrium at $x^* = \frac{1}{2}$.

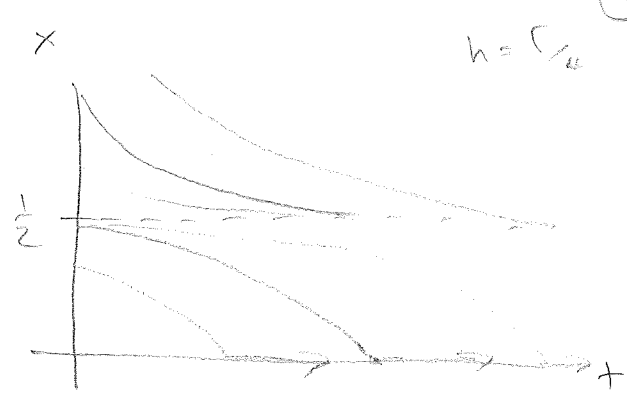
If $h > \frac{r}{4}$, we have no equilibria.

This model has some satisfying interpretations: If we don't fish, the population grows to its carrying capacity. If we fish a little bit, the carrying capacity is reduced. If we fish too much, though, or if the population starts too small, then this model has a large flaw – the population becomes negative.

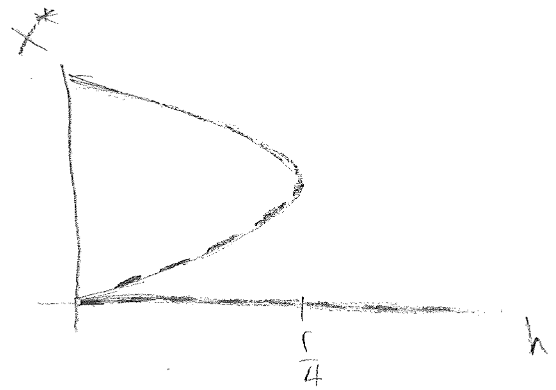
To fix this, we will modify our assumptions slightly (hopefully in a reasonable manner). We will assume that H fish per year are caught, unless $N<H$, in which case all the fish are caught and the population goes extinct. It's a little difficult to write this model down, but the solutions are easy to graph.

Constant rate Harvest:





We can summarize this information in a bifurcation diagram:



Important to note: If h increases a little bit, and is small x^* decreases a little bit. But if h is big enough and then increases even more, x^* suddenly drops from $\frac{1}{2} \to 0$ — The population collapses

Constant Effort harvest:

Now suppose we sell permits for T days per year. We will assume that fishermen put in the same amount of effort every day but that it's easier to catch fish when there are more available. That is, we let

$H(N) = qTN$, where q is the "catchability coefficient" – ie, the # of fish caught per fishing day per available fish.

Our model is therefore

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qTN.$$

If we let $x = \frac{N}{K}$ and $h = qT$, then

$$K\dot{x} = rKx(1-x) - hKx$$

$$\Rightarrow \boxed{\dot{x} = rx(1-x) - hx} = rx - hx - rx^2$$

The RHS is $rx(1-x) - hx = x \cdot (r - h - rx)$

Therefore, we have equilibria at

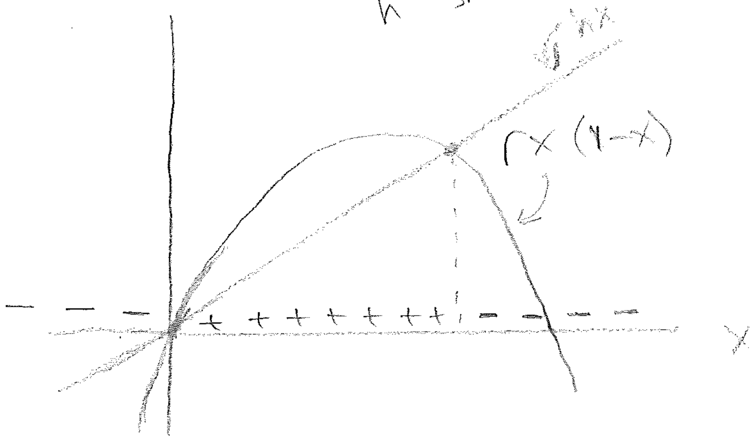
$x^* = 0$ and $r - h - rx^* = 0 \Rightarrow x^* = \frac{r-h}{r} = 1 - \frac{h}{r}.$

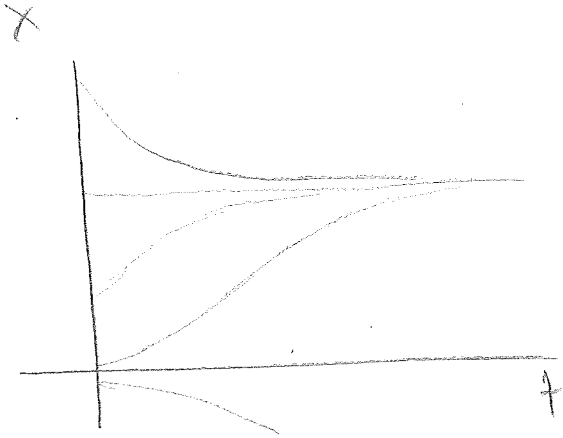
Are these equilibria stable? First, we'll decide graphically, then we'll check analytically.

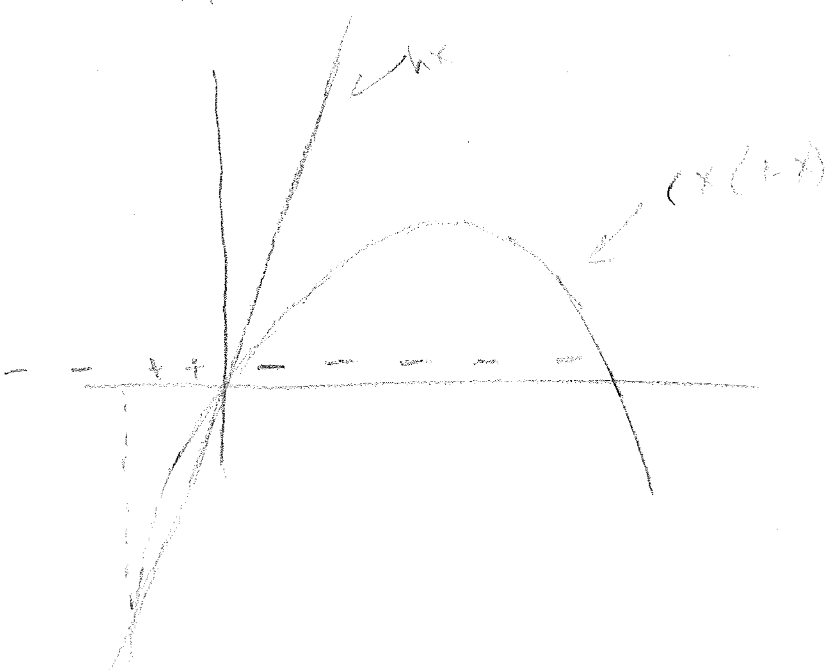
Constant Effort Harvest:

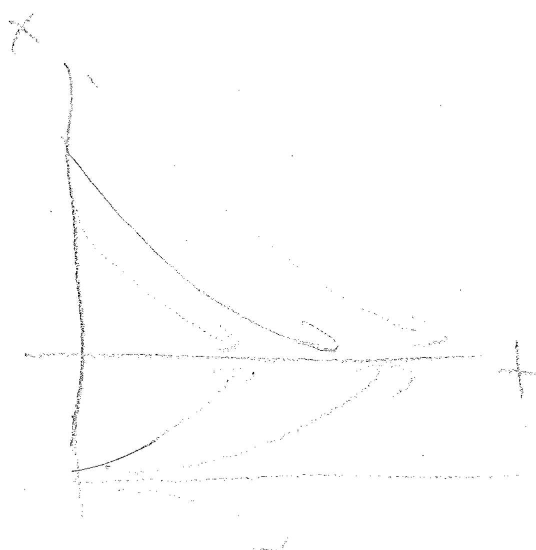
We could graph $rx(1-x)-hx$, but its usually easier to graph each term seperately.

h small

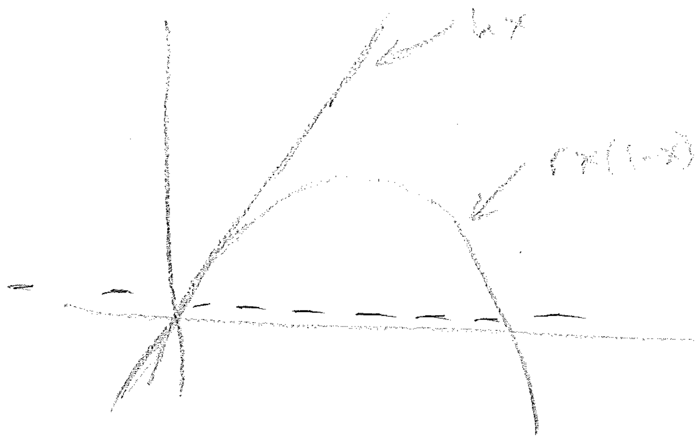


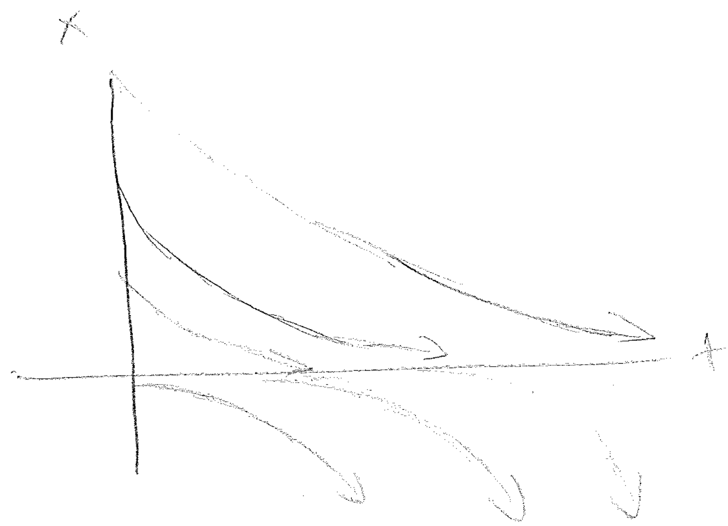






h just right





Check stability

Now we can check stability analytically:

$\dot{x} = rx(1-x) - hx = f(x)$

$f'(x) = r - h - 2rx$

$f'(0) = r - h$ If $r > h$, then $r - h > 0$, so $x^* = 0$ is unstable

If $r < h$, then $r - h < 0$, so $x^* = 0$ is stable

$f'\left(\frac{r-h}{r}\right) = r - h - 2r\left(\frac{r-h}{r}\right) = r - h - 2r + 2h = h - r$

If $r < h$, then $h - r > 0$, so $x^* = 1 - \frac{h}{r}$ is unstable

If $r > h$, then $h - r < 0$, so $x^* = 1 - \frac{h}{r}$ is stable

Bifurcation diagram:

Maximum Yield:

Important question: How many fish can we harvest?

Alternatively, how many permits should we sell to maximize harvest?

In the constant harvest model, we can catch H fish per year, as long as the population doesn't collapse, so we maximize yield by choosing

$$h = \frac{r}{4} \Rightarrow \boxed{Y_{max} = \frac{rK}{4}}$$

In the constant effort model, we can catch qTN fish per year. Assuming the population is usually around N^*, the stable equilibrium, we have

$$Y = qTN^* = qTK\left(1 - \frac{qT}{r}\right).$$

$$\left(x^* = 1 - \frac{h}{r} \Rightarrow N^* = Kx^* = K\left(1 - \frac{h}{r}\right) = K\left(1 - \frac{qT}{r}\right) \right)$$

We can only control T, so we want to maximize $Y(T)$.

$$Y(T) = qTK - \frac{q^2T^2K}{r}$$

$$\Rightarrow \frac{dY}{dT} = qK - \frac{2q^2K}{r}T = 0$$

$$\Rightarrow \frac{2q^2K}{r}T = qK$$

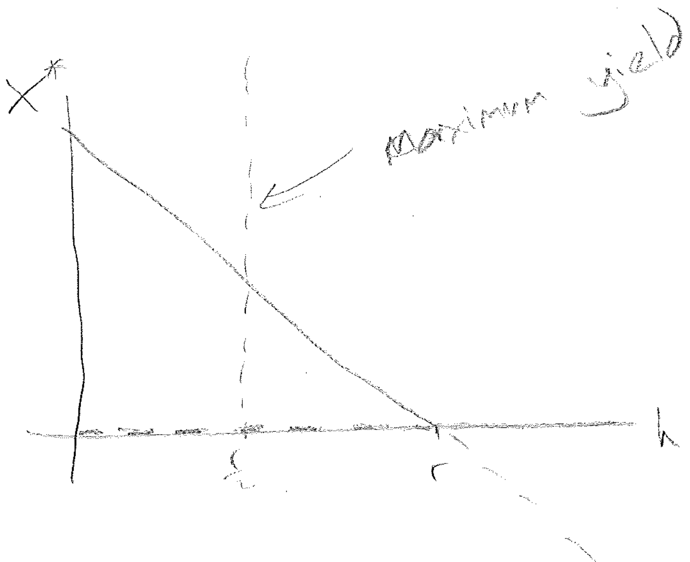
$$\Rightarrow T = \frac{qKr}{2q^2K} = \frac{r}{2q}$$

$$\Rightarrow Y_{max} = q \cdot \frac{r}{2q} \cdot K\left(1 - \frac{q \cdot \frac{r}{2q}}{r}\right) = \frac{rK}{2} \cdot \left(1 - \frac{1}{2}\right) = \boxed{\frac{rK}{4}}$$

Comparison:

These two plans both have the same yield. Which would you choose and why?

Constant H model



Constant Effort model