Example:

The American Eel is a threatened species in the East U.S. that used to be an extremely important food source for Native Americans. They spend most of their lives in freshwater rivers, then migrate out to the Sargasso Sea to spawn and die. It is reasonable to assume that population density affects the eel population's growth rate, but that density during the [illegible] [illegible] stages are far more important than spawning density.

As a first [illegible] [illegible] [illegible] [illegible] [illegible] the population [illegible] [illegible] [illegible] [illegible] [illegible], but with [illegible] [illegible] [illegible] [illegible]; instead of using

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right), \quad \text{we will use}$$

$$\frac{N_{t+\Delta t} - N_t}{\Delta t} = r N_t \cdot \left(1 - \frac{N_t}{K}\right)$$

$$\Rightarrow N_{t+\Delta t} = N_t + r\,\Delta t\, N_t \cdot \left(1 - \frac{N_t}{K}\right)$$

We will measure time in generations, so $\Delta t = 1$.
Therefore,

$$N_{t+1} = N_t + r N_t \cdot \left(1 - \frac{N_t}{K}\right)$$
$$= (1+r) N_t - \frac{r}{K} N_t^2 .$$

This means that the number of offspring per eel that was spawned last generation (not necessarily alive now) is

$$\frac{N_{t+1}}{N_t} = (1+r) - \frac{r}{K} N_t$$

which decreases linearly, like in the original logistic model.

To make our lives easier, let

$$x_t = \frac{r}{1+r} \frac{N_t}{K}, \quad \text{so} \quad x_{t+1} = \frac{r}{1+r} \frac{N_{t+1}}{K}$$

and $\mu = 1+r$

$$\Rightarrow \quad N_t = \frac{K \cdot \mu}{r} x_t$$

$$\Rightarrow \quad \frac{K\mu}{r} x_{t+1} = \frac{K\mu}{r} x_t + r \cdot \frac{K\mu}{r} x_t \left(1 - \frac{K\mu}{r} \frac{r x_t}{K}\right)$$

$$\Rightarrow \quad x_{t+1} = x_t + r x_t \cdot (1 - \mu x_t)$$

$$\Rightarrow \quad x_{t+1} = (1+r) x_t - \mu x_t^2 = \mu x_t (1 - x_t)$$

This model is named, appropriately enough,

$$x_{t+1} = \mu x_t (1 - x_t) = f(x_t)$$

the logistic map.

A few preliminary points. If $x_t > 1$, then $x_{t+1} < 0$, which doesn't really make sense for a population. We will therefore insist that

$0 \le x_t \le 1$.

If $0 \le \mu \le 4$, then $f(x_t) \le 1$, so $x_{t+1} \le 1$. This means that, as long as we require $\mu \in [0, 4]$ and $x_0 \in [0, 1]$, then x_t will always be in $[0, 1]$, and so always positive.

Analysis: What are the equilibria of the logistic map?

$$f(x^*) = x^*$$

$$\Rightarrow \mu x (1 - x) = x$$

$$\Rightarrow \mu x (1 - x) - x = 0$$

$$\Rightarrow x(\mu(1 - x) - 1) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad \mu(1 - x) = 1$$

$$\Rightarrow x = 1 - \frac{1}{\mu}$$

$(N^* = K)$

$$1 - \frac{1}{1+r} = \frac{r}{1+r} \frac{N}{K}$$

$$\Rightarrow 1 + r - 1 = r \frac{N}{K}$$

$$\Rightarrow r = r\frac{N}{K}$$

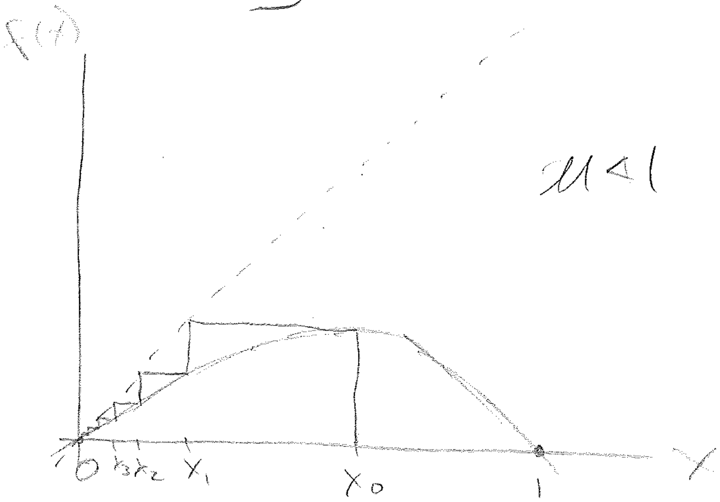
$$\Rightarrow N = K$$

$x^* = 0$ or $x^* = 1 - \frac{1}{\mu}$

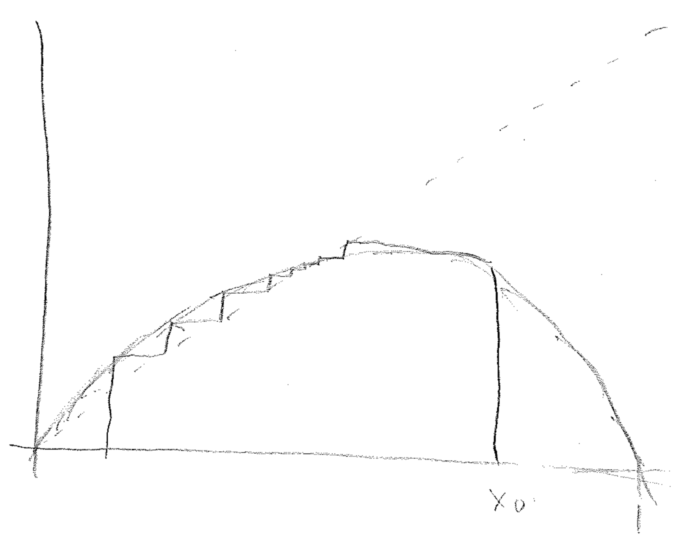
We restricted $x \in [0,1]$, so $x^* = 1 - \frac{1}{\mu}$ is only feasible if $1 - \frac{1}{\mu} > 0 \Rightarrow \mu > 1$.

Remember, $\mu = 1 + r$, so $\mu > 1 \Rightarrow r > 0$.
This means we only have a non-zero equilibrium if the growth rate is positive.

Stability:



$x^* = 0$ is the only equilibrium, and it looks stable

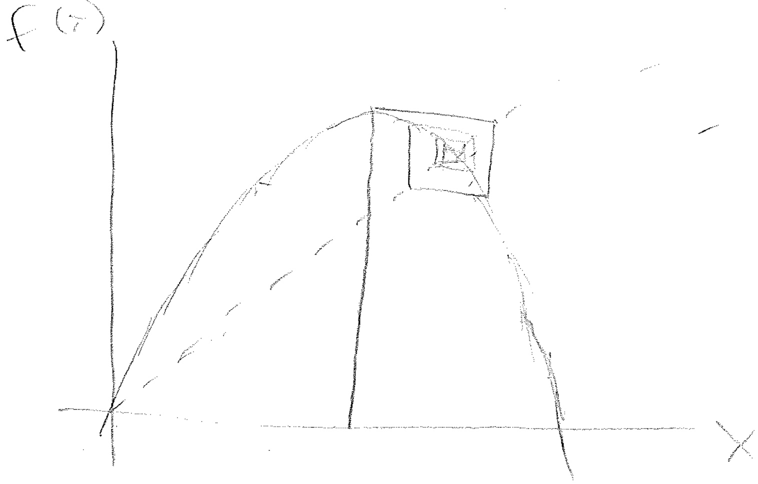


μ slightly bigger than 1

(e.g. $\mu = 1.5$)

$x^* = 0$ is unstable, $x^* = 1 - \frac{1}{\mu}$ is stable Node

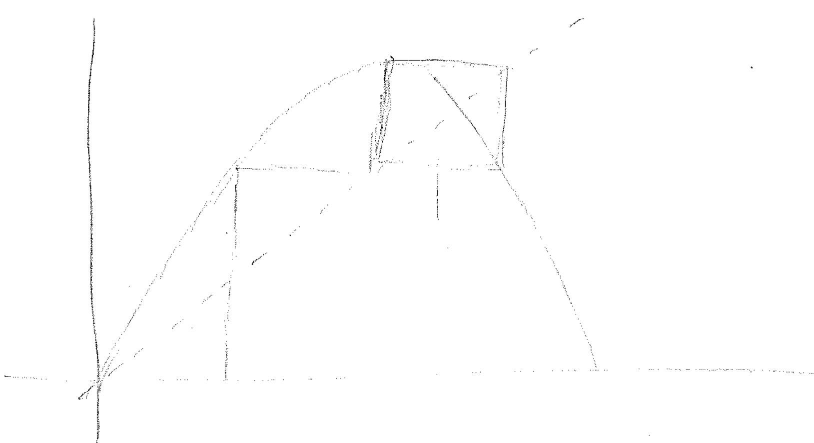
Larger μ (e.g. $\mu = 2.8$)



$x^* = 0$ unstable;

$x^* = 1 - \frac{1}{\mu}$ stable spiral

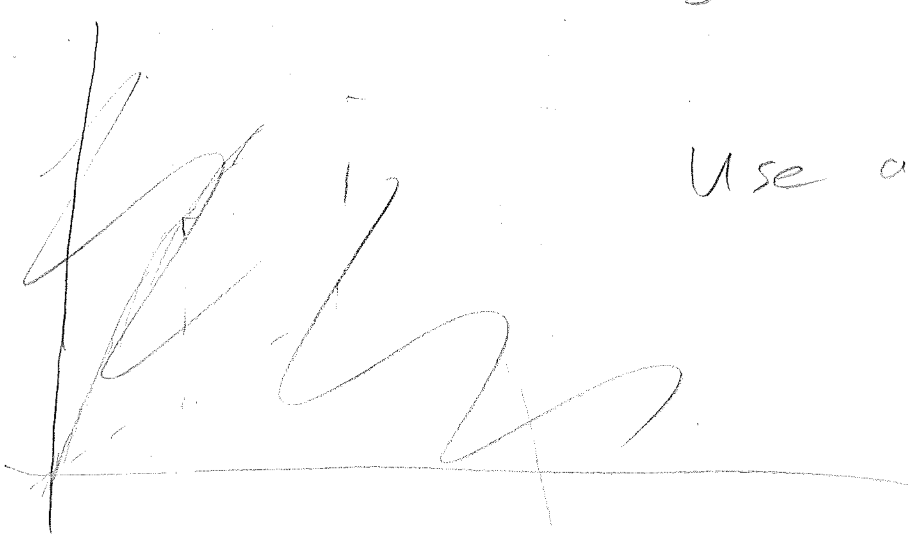
Larger μ (e.g. $\mu = 3.1$)



$x^* = 0$ unstable,

$x^* = 1 - \frac{1}{\mu}$ unstable spiral, but there is a "stable 2-cycle"!

Even Larger μ (e.g. $\mu = 3.5$)



Use applet

can get a stable 4-cycle.

Even larger μ (e.g. $\mu = 3.8$)

lead to chaos - No stable orbits

We want to attempt to find these cases analytically.

First, let's look at fixed point stability:

$$x_{t+1} = \mu x_t (1 - x_t) = f(x_t)$$

Remember, stability of x^* given by:

x^* is stable if $|f'(x^*)| < 1$

x^* is unstable if $|f'(x^*)| > 1$

x^* is node if $f'(x^*) > 0$

x^* is spiral if $f'(x^*) < 0$

$x^* = 0$: $\quad f'(x) = \mu - 2\mu x$

$$\Rightarrow f'(0) = \mu$$

This is always a node, because $\mu > 0$. It is stable if $\mu < 1$ and unstable if $\mu > 1$.

$x^* = 1 - \frac{1}{\mu}$:

$$f'(x) = \mu - 2\mu x$$

$$\Rightarrow f'\left(1-\frac{1}{\mu}\right) = \mu - 2\mu \cdot \left(1 - \frac{1}{\mu}\right)$$

$$= \mu - 2\mu + 2$$

$$= 2 - \mu$$

- If $\mu < 1$, $2 - \mu > 1$, so
  x^* is an unstable node
- If $1 < \mu < 2$, $0 < 2 - \mu < 1$, so
  x^* is a stable node
- If $2 < \mu < 3$, $-1 < 2 - \mu < 0$
  x^* is a stable spiral
- If $\mu > 3$, x^* is an unstable spiral

Start of bifurcation diagram: