Overview:

We have been looking at a model of eel population dynamics using the logistic map:

$$x_{t+1} = \mu x_t (1 - x_t)$$

We found that there are two equilibria, $x^* = 0$ and $x^* = 1 - \frac{1}{\mu}$.

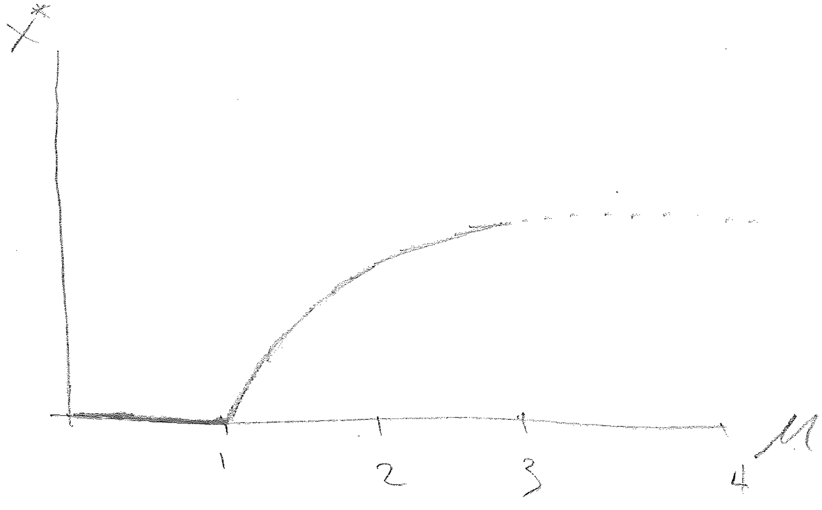
Zero is always a node, and it is stable for $0 < \mu < 1$ (when the growth rate is negative) and unstable otherwise.

$x^* = 1 - 1/\mu$ is an unstable node for $0 < \mu < 1$ (and corresponds to a negative population).

For $\mu > 1$, $x^* = 1 - 1/\mu > 0$, and it is:

- A stable node for $1 < \mu < 2$
- A stable spiral for $2 < \mu < 3$
- An unstable spiral for $\mu > 3$.

We can summarize this information in a bifurcation diagram.



We also found graphically that higher values of μ led to cycles. In particular, we used cobweb diagrams to find stable 2-cycles and stable 4-cycles. Finally, we saw that even higher values of μ lead to aperiodic (chaotic) dynamics.

Today, we want to find some of these cycles.

How do we define a cycle? Remember that an equilibrium is a value of x_t such that

$$x_{t+1} = f(x_t) = x_t.$$

That is, x at the next time step is the same as x at this time step.

A 2-cycle is a pair of values x_t such that

$$x_{t+2} = x_t.$$

That is, x two time steps from now is the same as x now. Why is there a pair of values? If $x_t = x_{t+2}$, then $x_{t+1} = x_{t+3}$.

In general, we define an N-cycle as N values x_t such that $x_{t+N} = x_t$. Notice that an equilibrium is a 1-cycle.

Notice that, if $x_{t+1} = f(x_t)$, then

$$x_{t+2} = f(x_{t+1}) = f(f(x_t))$$

$$x_{t+3} = f(x_{t+2}) = f(f(f(x_t)))$$

$$\vdots$$

$$x_{t+N} = f(x_{t+N-1}) = \underbrace{f(f(\cdots f}_{N \text{ times}}(x_t))))$$

We will write this as $x_{t+N} = f^N(x_t)$.

(Note that f^N is not "f to the Nth power".)

This means that, if we want to find a 2-cycle, we need to look at the map

$$x_{t+2} = f^2(x_t)$$

Although it looks more complicated, this is really just a normal map. The 2 is not special. In particular, if we change our units of time to $\tau = \frac{t}{2}$ and let $F = f^2$, then we have

$$x_{\tau+1} = F(x_\tau).$$

An equilibrium (1-cycle) of this map is a 2-cycle of the original map.

$$f^2(x) = f(f(x)) = f(\mu x(1-x))$$

$$= \mu(\mu x(1-x))(1 - \mu x(1-x)) = F(x)$$

$$= (\mu^2 x - \mu^2 x^2)(1 - \mu x(1-x))$$

$$= (\mu^2 x - \mu^2 x^2)(1 - \mu x + \mu x^2)$$

$$= \mu^2 x - \mu^3 x^2 + \mu^3 x^3 - \mu^2 x^2 + \mu^3 x^3 - \mu^3 x^4$$

$$= -\mu^3 x^4 + 2\mu^3 x^3 - (\mu^3 + \mu^2)x^2 + \mu^2 x.$$

Equilibria occur when $F(x)=x$, so

$$-\mu^3x^4+2\mu^3x^3-\mu^3x^2-\mu^2x^2+\mu^2x=x \quad (1)$$

$$\Rightarrow \mu^3x^4-2\mu^3x^3+\mu^3x^2+\mu^2x^2-\mu^2x+x=0$$

This is a quartic, so we would normally be out of luck, but we actually already know two factors. Why? Because an equilibrium of F is also a 2-cycle of f. (If $x_{t+1}=x_t$, then $x_{t+2}=x_t$ as well.) This means that $x=0$ and $x=1-1/\mu$ are solutions to (1), so we can factor out an x and an $x-1+1/\mu$. Actually, we will factor out $\mu x-\mu+1$ to make life easier.

$$\Rightarrow x\cdot\left[\mu^3x^3-2\mu^3x^2+\mu^3x+\mu^2x-\mu^2+1\right]=0$$

Remember how to divide polynomials?

(Neither did I, but this is good practice.)

$$\mu^2x^2 - (\mu^2+\mu)x + (\mu+1)$$

$$\mu x - \mu + 1 \;\big)\; \mu^3x^3 - 2\mu^3x^2 + (\mu^3+\mu^2)x + (1-\mu^2)$$

$$\mu^3x^3 - \mu^3x^2 + \mu^2x^2$$

$$-\mu^3x^2 - \mu^2x^2 + (\mu^3+\mu^2)x$$

$$-\mu^3x^2 - \mu^2x^2 + \mu^3x + \mu^2x - \mu^2x + \mu x + 1 - \mu^2$$

$$\mu^2x + \mu x$$

$$\mu^2x + \mu x - \mu^2 + \mu - \mu + 1$$

$$0$$

$$\Rightarrow \quad x \cdot (\mu x - \mu + 1) \cdot \left(\mu^2x^2 - (\mu^2+\mu)x + (\mu+1)\right) = 0$$

We can solve the rest with the quadratic equation. We get

$$\left.\begin{array}{l} x = 0 \\ \mu x - \mu + 1 = 0 \Rightarrow x = 1 - 1/\mu \end{array}\right\} \text{equilibria}$$

$$\mu^2x^2 - (\mu^2+\mu)x + (\mu+1) = 0$$

$$\Rightarrow x = \frac{\mu^2+\mu \pm \sqrt{(\mu^2+\mu)^2 - 4\mu^2(\mu+1)}}{2\mu^2}$$

$$= \frac{\mu(\mu+1) \pm \sqrt{\mu^2(\mu+1)^2 - 4\mu^2(\mu+1)}}{2\mu^2}$$

$$= \frac{\mu(\mu+1) \pm \sqrt{\mu^2(\mu+1)[\mu+1-4]}}{2\mu^2}$$

$$= \frac{\mu+1 \pm \sqrt{(\mu+1)(\mu-3)}}{2\mu}$$

How many solutions are there?

If $\mu > 3$, then the discriminant is positive, so there are two 2-cycles.

If $\mu = 3$, there is one 2 cycle.

If $\mu < 3$, there are no 2 cycles.

We also need to check their stability.

To do this, we need to find $F'(x^*)$. We could do this by brute force, but it gets ugly. Instead, note that

$F(x) = f(f(x))$. Let x_1 and x_2 be our solutions from above. We know $f(x_1) = x_2$ and $f(x_2) = x_1$ (because they are in a 2-cycle).

We have

$$F'(x_1) = \frac{d}{dx}\left(f(f(x))\right)_{x=x_1}$$

$$= f'(f(x_1))\cdot f'(x_1)$$

$$= f'(x_2)\cdot f'(x_1).$$

Likewise,

$$F'(x_2) = f'(f(x_2))\cdot f'(x_2)$$

$$= f'(x_1)\cdot f'(x_1)$$

Since these are the same, the cycles flip stability at the same time.

$f'(x) = \mu - 2\mu x$, so

$$F'(x_1) = (\mu - 2\mu x_1)(\mu - 2\mu x_2)$$

$$= \mu^2\cdot(1-2x_1)(1-2x_2)$$

$$= \mu^2\cdot\left(1 - 2(x_1+x_2) + 4x_1x_2\right)$$

$$= \mu^2 \cdot \left[1 - 2\left(\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu} + \frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu} \right) \right.$$

$$\left. + 4 \cdot \left(\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu} \right) \cdot \left(\frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu} \right) \right]$$

$$= \mu^2 \cdot \left[1 - \frac{2(\mu+1)}{\mu} + \frac{(\mu+1)^2 - (\mu+1)(\mu-3)}{\mu^2} \right]$$

$$= \mu^2 \cdot \left[1 - \frac{2(\mu+1)}{\mu} + \frac{(\mu+1)\cdot(\mu+1-\mu+3)}{\mu^2} \right]$$

$$= \mu^2 - 2\mu(\mu+1) + (\mu+1)\cdot 4$$

$$= -\mu^2 + 2\mu + 4.$$

This means that our 2-cycles are stable if

$$\left| -\mu^2 + 2\mu + 4 \right| < 1$$

To find these boundaries, we solve

$$4 + 2\mu - \mu^2 = 1 \quad \text{and} \quad 4 + 2\mu - \mu^2 = -1$$

$\Rightarrow \mu^2 - 2\mu - 3 = 0$

$(\mu - 3)(\mu + 1) = 0$

$\Rightarrow \boxed{\mu = 3}$ or -1

$\mu^2 - 2\mu - 5 = 0$

$\Rightarrow \mu = \dfrac{2 \pm \sqrt{4 + 20}}{2}$

$= \dfrac{2 \pm 2\sqrt{6}}{2}$

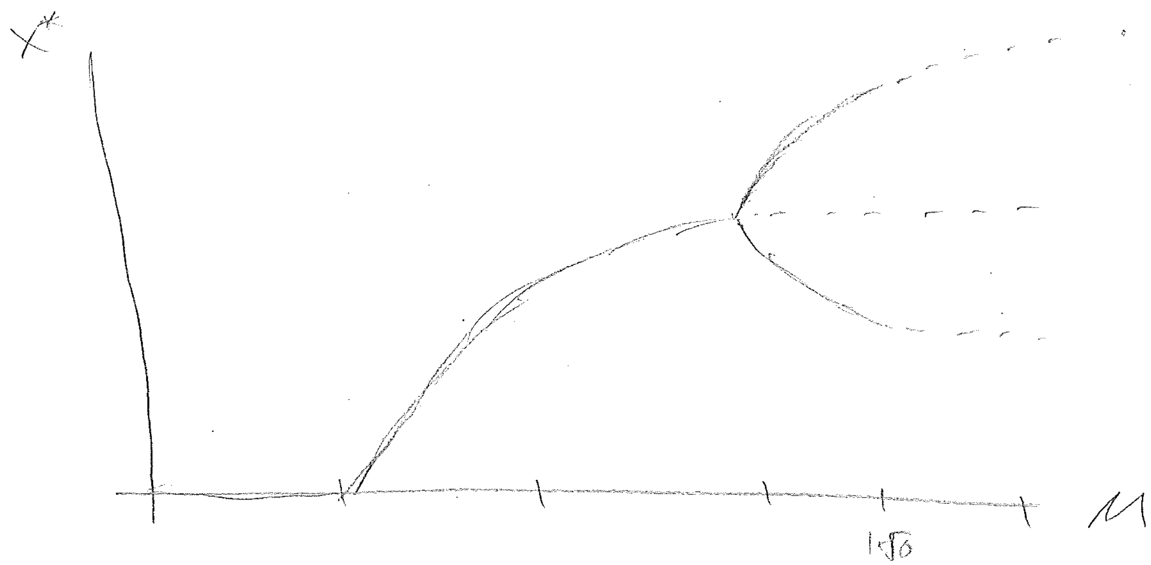
$= 1 \pm \sqrt{6}$

$\boxed{\mu = 1 + \sqrt{6}} \approx 3.45$

Our 2-cycles are stable between

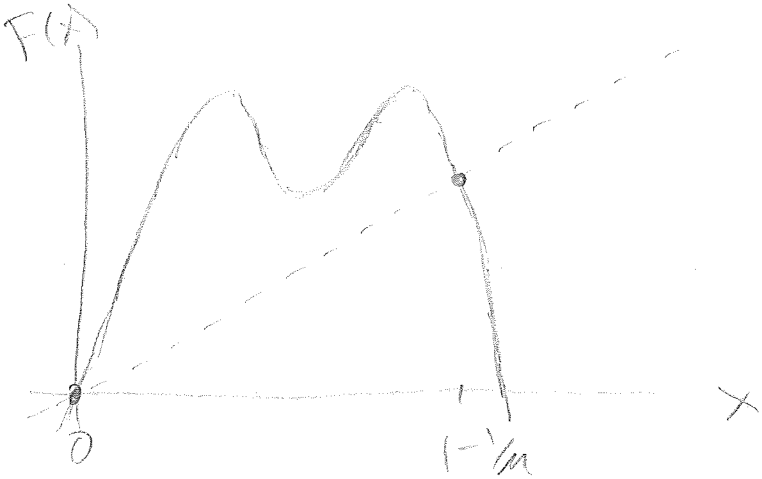
$$3 < \mu < 1 + \sqrt{6},$$

and unstable otherwise. We can therefore expand our bifurcation diagram:

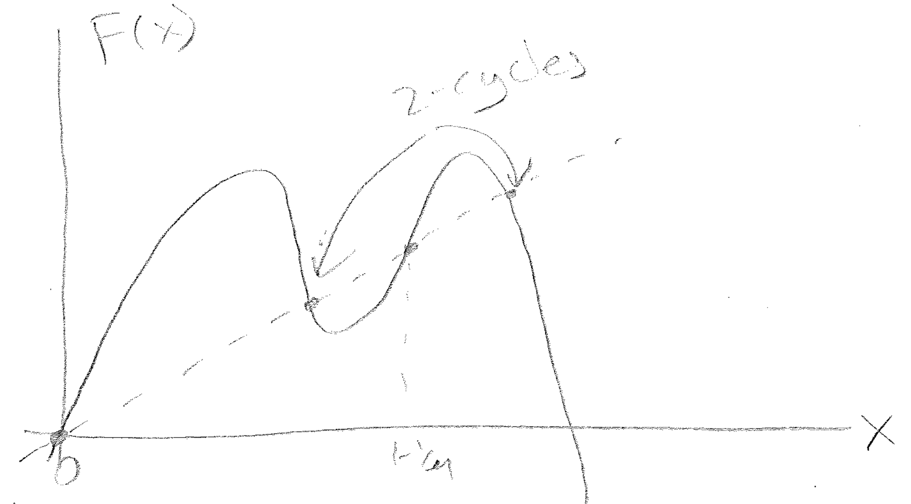


We can also look at these cycles graphically

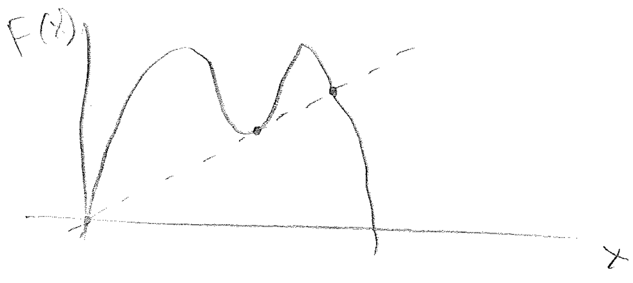
$F(x)$ is a quartic and looks like:



When μ is small ($\mu<3$), this graph only crosses the 45 degree line twice - at the two original equilibria. As μ gets bigger ($\mu>3$) we get 4 intersections. (The equilibria + the 2-cycles)



At exactly $\mu=3$, the curve is tangent to the 45° line.



We can try to keep going and find larger cycles, but there isn't much hope. $f^N(x)$ is a $(2^N)^{th}$ degree polynomial. To find 3-cycles, we would need to factor a 8^{th} degree polynomial. We already know two (the equilibria) but that still leaves a 6^{th} degree, which we can't solve. (You can actually find where the 3-cycles start, but it's still quite difficult.)

Similarly, finding 4-cycles requires factoring an 16^{th} degree polynomial. We know 4 factors (the equilibria and 2-cycles) but that still leaves a 12^{th} degree. Again, we can find the μ where 4-cycles appear, but finding them in general is impossible. Higher cycles just get harder. In general, people resort to numerics after 2-cycles.